SYNTHESIS OF "OPTIMUM" TRANSIENT RESPONSE — CRITERIA AND STANDARD FORMS

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FOREWORD

This report was prepared as a portion of the project identified
by Research and Development Order Number R-206-11 with the title
"Design and Development of a Landing System Evaluator". The project
is conducted by the All-Weather Section, Engineering Branch, Direct-
orate of Flight and All-Weather Testing with Major Richard C. Lathrop
as project engineer.

ABSTRACT

Various methods for synthesizing servomechanisms are reviewed, and it is pointed out that, in general, criteria for determining an optimum solution to the synthesis problem are vague. Stability criteria, frequency, and root locus methods reduce to conditions on the transfer function constants which may be made mathematically specific following Whiteley's suggestion of "standard forms".

Eight mathematical criteria for optimum transient responses are critically examined, and the clear superiority of the minimum integral of time multiplied absolute value of error ($\int_0^\infty t|e|\,dt$) is demonstrated.

Tables of "standard forms" for optimum zero-displacement error systems are presented through the eighth order, and standard forms for zero-velocity and zero-acceleration-error systems are presented through the sixth order. Applications to the design of duplicator servomechanisms, pulse amplifiers, and servo systems operating on noisy inputs are pointed out.

An Appendix contains a discussion of computer techniques including the absolute value unit and the generation of error responses by an extension of Beck's method.

PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDER:

HUGH B. MANSON, JR
Colonel, USAF
Director of Flight and All-Weather Testing

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INTRODUCTION

Transient behavior is an attribute of many measurement, control, and communication devices. Engineering design of such physical systems often involves the choice of design variables which will insure optimum transient behavior. Mathematical techniques applicable to this engineering design problem have been rapidly developed and extended during the past ten years. The lack of a practical mathematical definition of optimum transient behavior, however, has forced empiricism into the design procedures, and the procedures themselves, for the most part, are esoteric and laborious. Development of a unitary figure of merit for transient behavior and tabulation of optimum system characteristics would be a boon to the designer.

The mathematical methods applied to the problem of optimizing the transient behavior of physical systems depend uniformly on the conception of the <u>response of a transfer system</u>.[1] A transfer system which is characterized by an <u>input and a related output</u> is illustrated in Figure 1. The physical system, which might be an electric circuit, a mechanical device, a vehicle, or a process, is shown in a block connecting the input to the output. This block may be conceived of as an operator which metemorphoses a given input, $r(t)$, to produce a unique output, $c(t)$. The output is said to be the <u>response</u> of the system to the given input. The most fundamental relationship which gives the dependence of the output (effect) on the input (cause) is usually the differential equation which describes the physical system. In the case in which the differential equation is linear with constant coefficients, there are several possible mathematical descriptions, based on operational methods.[2,3] A linear transfer system may be defined alternatively by an explicit differential equation, a <u>transfer function $\frac{C(s)}{R(s)}$</u>, a <u>weighting function</u>, $w(t)$, a <u>frequency response function $\frac{C(j\omega)}{R(j\omega)}$</u>, or by its <u>poles</u> and <u>zeros</u>. All of these mathematical forms are equivalent in the sense that each represents the physical system, and it is possible to obtain one representation from another by mathematical operations.

A simple positioning servomechanism, its linear second order equation, and its transfer function are illustrated in Figure 2. The weighting function, which is the unit impulse response, its integral, the step function response, and the response to a sinusoid are also shown. The response to a sinusoid at a number of frequencies is the frequency response function. The denominator polynomial of the transfer function, equated to zero, is called the <u>characteristic equation</u> of the system. The roots of the characteristic equation are called the <u>poles of</u> the system, and the roots of the numerator polynomial, equated to zero, are called the <u>zeros</u> of the system.

Transfer functions of the linear systems considered in this paper can always be reduced to the normalized form

$$\frac{C(s)}{R(s)} = \frac{p_m s^m + \dots + p_2 s^2 + p_1 s + p_0}{s^n + q_{n-1} s^{n-1} + \dots + q_2 s^2 + q_1 s + 1} \tag{1}$$

The procedure for deriving the normalized transfer function from the system differential equation is outlined in Appendix II.

SYNTHESIS METHODS

Three distinct problems occur in the application of differential equations and operational mathematics to physical systems. These problems have been named the <u>analysis</u>, <u>instrument</u>, and <u>synthesis</u> problems. The analysis problem is: given the input and the mathematical description of the system, find the output. The instrument problem is: given the output and the mathematical description, find the input. Synthesis of a linear transfer system has been defined as the following problem: given the input and the desired output, determine the mathematical description.

It is clear that the synthesis problem is intimately related to engineering design. Typical inputs are often known and the "desired" output is often subject to specification. The mathematical description of the required system is a preliminary to its physical realization. The realization of the physical system is excluded from consideration in this paper, however, and attention will be confined to what Bubb[4] has termed the "mathematical attorney" of the system.

Attention will be further confined to the class of linear transfer systems called "duplicators", i.e., those systems in which the shape of the output approximately duplicates the shape of the input. This class, however, is very broad and covers many open and closed loop control systems, servomechanisms, regulators, filters, amplifiers, and instruments.

The synthesis of duplicators has suffered, generally, from an inadequate definition of "desired output". The desired output of a duplicator usually is perfect reproduction of the shape of the input, but this is physically impractical. The desired output is, therefore, often formulated rather loosely on a frequency response basis in terms of gain margin and phase margin, or on a transient basis in terms of intuitive concepts of rise time and overshoot.

The first and most obvious condition on the output is stability. The output of a duplicator is unequivocally desired to be stable. Fortunately stability can be precisely defined, and conditions for stability can be rather easily calculated. The characteristic equation of a linear transfer system completely determines its stability. In fact, a necessary and sufficient condition which precludes instability is that the real roots and the real parts of the complex roots of the system characteristic equation are not positive. The roots of the characteristic equation depend on the coefficients of the equation. There are, therefore, certain conditions on the coefficients of the characteristic equation which must be satisfied in order to insure that the system is stable.

Consider the normalized characteristic equation

$$s^n + bs^{n-1} + cs^{n-2} + \ldots + ps^2 + qs + s^0 = 0 \qquad (2)$$

None of the coefficients, b, c, . . ., may be zero and all of them, including the initial and final unit coefficients, must have the same sign in order for the system to be stable. In addition, the functional relations between the coefficients defined by the Routh-Hurwitz criterion must hold true. These latter test functions are tabulated for the various orders of non-dimensional characteristic equations through the eighth order in Table 1. Theoretically, n-1 test functions have to be applied to each characteristic equation. It has been shown, however,

by Frazer and Duncan[5] that the conditions tabulated are practically all that are necessary since they will <u>first</u> indicate the change in the character of the roots on going from a stable to an oscillatorily divergent system. The change on going from a stable system to an aperiodically divergent one is first indicated by a change in the sign of the last coefficient.

In addition it may be noted that in an equation of order "n", such as equation 2, the negative sum of all the real roots and the real parts of the complex roots is the coefficient "b" of the n-1 power of the variable and the product of all the roots is the constant times $(-1)^n$. This is the basis of some approximate factorization methods, and further illustrates the fact that conditions on the roots may be considered to be conditions on the coefficients.

A method much in vogue among aerodynamicists for the approximate synthesis of airplane dynamics involves plotting the Routh-Hurwitz test function, equated to zero, as a function of two design variables with all others held fixed. This gives a graphical representation, called a stability diagram, of the boundary between stable and unstable combinations of the two selected variables. An extension of this method due to W. Brown[6] enables one, at the cost of considerable labor, to superimpose lines of constant oscillatory period and lines of constant logarithmic decrement (damping) on such a plot. The degree of stability provided by a selected combination may be determined in this fashion. Stability diagrams for third and fourth order systems are shown in Figures 3 and 4. Figure 3 may be used for the solution of the cubic equation since the real and imaginary parts of the complex pair of roots are available directly, and the third root is the negative inverse of $\alpha^2 + \beta^2$.

In the case of closed loop systems, such as servomechanisms and feedback amplifiers, the Nyquist Criterion is often applied to determine stability.[7] The complete and mathematically rigorous statement of the criterion simply expresses the fact that, with a sinusoidal input, the signal which is fed back must not arrive at the input end of the transfer system with an in-phase component larger than unity. If it does, the feedback signal will augment the input signal and the output of the system will diverge. For transfer systems which are stable under open loop conditions, the Nyquist Criterion may be applied by plotting the open loop frequency response, $\frac{C(j\omega)}{R(j\omega)}$, on a polar diagram. Whether or not the -1 + j^0 point is encircled by the complex frequency response function as the frequency is varied from zero to infinity determines the stability of the system.

As in the case of stability diagrams, it is possible to extend the technique of mapping the complex frequency function to the determination of the degree of stability; in particular, to the determination of logarithmic decrements and damping ratios.

The shape of the complex frequency response function

$$\frac{C(j\omega)}{R(j\omega)} = \left[\frac{P_m \lambda^m + P_{m-1} \lambda^{m-1} + \ldots + P_1 \lambda + P_0}{\lambda^n + Q_{n-1} \lambda^{n-1} + \ldots + Q_2 \lambda^2 + Q_1 \lambda + Q_0} \right]_{\lambda = j\omega} \quad (3)$$

is, of course, dependent on the various coefficients of the numerator and denominator polynomials. It appears that, as in the case of the Routh-Hurwitz Criterion, the stability and degree of stability of the system depend on certain relations between the coefficients of the transfer function.

The synthesis of servomechanisms is more often carried out using the logarithmic plots developed by Bode[8]. Since the phase angle of the frequency response function was shown by him to be related to the rate of change of the amplitude function, it is often possible to accomplish much of the synthesis procedure graphically using straight line asymtotic approximations. In combination with Nichols' method for obtaining the closed loop response of a servomechanism, this analysis method is a simple and powerful one.[2] The synthesis process is carried out by analyzing the effects first on the open, then on the closed loop frequency response, of various possible system changes which might give an acceptable overall performance. The rules governing maximum amplitude ratio, gain margin, phase margin, and the length of asymptotes between break points, if applied with care, may result in a system with adequate transient performance. The outcome of the analysis, however, is not necessarily the optimum system, and the result indicates only approximately the changes in the system which would bring about improved transient performance. All this is in decided contrast to the situtation usually facing an amplifier or filter designer, to whom the frequency response is all important and is susceptible to precise specification.

The root locus method developed by Evans[9,10] is also useful in the synthesis of closed loop systems. In this method the varying position of the poles and zeros of the system transfer function are plotted in the complex plane as a function of one loop gain. Both the frequency and transient responses of the system may be inferred from such a plot. There are, however, no generally known specifications for the optimum location of the poles and zeros beyond the bare specification of stability or degree of stability (damping ratio, time to damp). The methods for locating the poles so as to achieve suitable values of these quantities have been developed in an unpublished work by J.R. Moore. If it were possible to specify the optimum location of the poles and zeros, that would amount to the specification of the form and the coefficients of the system transfer function.

In the design of filters (or servomechanisms) operating on noisy inputs, the elegant mathematical methods of Wiener[11] may be applied. It is assumed that the power spectral densities of the signal input and noise are known. The desired output is precisely defined as the output which follows or anticipates the input but rejects the noise in such a way that the root mean square value of the error is the least. The result is an explicit mathematical statement of the desired weighting function for the system. Phillips[2] has simplified the application of this method to the synthesis of duplicator servomechanisms by assuming a form for the system transfer function and leaving only one parameter open to adjustment so as to obtain the minimum root mean square error.

For a few cases, the direct synthesis of transient response is possible. Charts showing the transient responses as functions of non-dimensional system parameters are given by G. Brown[13], Draper and Schliestett[12], Bretoi[14], and others for first and some second and third order systems. In very special cases, as in the synthesis of pulse amplifier interstage circuits, Wallman[15] has presented such charts for transient responses of higher ordered systems.

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The calculation of transient responses is a tedious procedure. Fortunately, in recent years, there have been put into operation many differential analyzers of various types. These devices take most of the labor out of the calculation of transient responses of physical systems, but in a system with a large number of parameters, the correct combination for good, stable performance may be difficult to find by cut and try methods.

Manger[2] has commented on the analogy between low pass filter characteristics and the characteristics of duplicator servomechanisms, and has pointed out that a system with a wide low pass band is likely to have a good transient response. In particular, he has illustrated the transient response of the so-called "ideal low pass filter". Further investigation has been made into the desirable and realizable transient responses of pulse amplifier interstage filters.[16,17,18] If the exact form of a known transient response is deemed suitable, new filters or servomechanisms may be synthesized by analogy to the physical system which has been analyzed. Note that this method of specifying the exact form of the transient response is the equivalent to the specification of a desired system transfer function!

Whiteley[19] has taken the latter step and has tabulated the coefficients of the polynomial denominators of system transfer functions for systems of various orders, and with three different kinds of numerators. He has named these explicit numerical functions for the polynomial denominators standard forms. Dynamic systems which are synthesized in accordance with the standard forms will have an output response to an input step as illustrated in the charts which accompany the standard forms. This is the desired output. The criteria which Whiteley used to judge the "goodness" of desired outputs were, in one case, a maximally flat frequency response, and in others, the magnitude of the peak overshoot.

Some of the other criteria of "goodness" for transient responses which have been applied by the authors of transient response charts and by users of such charts and of differential analyzers are: delay time, solution time, time to first zero, peak ratio, and overshoot. While suitable values for all these or other applicable quantities may be known in general, this knowledge usually does not give much insight into the most favorable adjustments to make to the system. In many cases where several figures of merit are applied, it is possible to "trade" some of one for a better value of another. The direct synthesis of transient responses, whether from charts, from Whiteley's standard forms or by means of a differential analyzer, is subject to the objection that the desired output has not been defined precisely enough.

Table II summarizes the various methods of synthesizing servomechanisms (and other dynamic systems) which have been discussed above, illustrates the criteria which are applied, and offers some comment on each method.[20,21]

CRITERIA FOR TRANSFER SYSTEM RESPONSE

The choice of specific design variables in the synthesis of duplicators depends completely on the criteria which are applied in judging how well the output follows the input. Speed and stability of response are desirable. These qualities may be indicated numerically by defining equivalent time constant, solution time, time to first zero, overshoot, or peak ratio, and so forth. A more fruitful approach to the problem, however, would be to develop a unitary figure of merit or criterion of goodness for the transient response which would take all or most of its characteristics into account.

Ideally, such a criterion should have three basic attributes: reliability, ready applicability, and selectivity. It should be reliable for a given class of systems, so that the user can apply it with confidence to any transfer system within the class, including those about which he has little a priori information. The criterion should be easy to apply. A graphical construction, an analytic expression, or the analog of a mathematical expression might be acceptable forms of a criterion. Finally, the criterion should be selective; that is, it should indicate a discernible difference between good systems and those which are not quite as good.

This paper is concerned with the transient performance (step function response) of servomechanisms or similar transfer systems. Only those systems which have a steady state displacement error of zero when subjected to an input step function are considered. An example of such a system would be a simple linear second order system which has the normalized transfer function

$$\frac{C(s)}{R(s)} = \frac{1}{s^2 + 2\zeta s + 1} \tag{4}$$

Figure 5(a) shows the error responses $e(t) = r(t) - c(t)$ for this system for various values of ζ, when $r(t)$ is a unit step function. A perfect response would be an output step function identical to the input with no error at any time. The actual responses of Figure 5(a) differ from this perfect response in various ways. In particular, certain characteristics of the responses provide a measure of the degree to which the responses approximate the ideal response. Three commonly used characteristics are (1) the time for the error (difference between input and output) to reach its first zero, (2) the amount of the first overshoot, expressed as a percentage of the initial error, and (3) the solution time (time for the error to reach and remain within 5% of its initial value). These three quantities are plotted in Figure 6 as functions of the damping ratio, ζ. It is clear that the percent overshoot and the time to first zero are conflicting characteristics, in the sense that their minimum values occur at different damping ratios. If these two characteristics of the simple second order responses were used as criteria, the design problem would consist of selecting that value of which affords the best compromise between small overshoot and fast rise time. On the other hand, the solution time can be used alone as a criterion of performance, since it combines, in a sense, the properties of the other two characteristics. The solution time is a minimum for a damping ratio of about 0.7 in a simple second order system. Its applicability is not restricted to linear second order systems, for it may be applied to higher order and non-linear systems as well.

The solution time criterion, applied to a linear second order system, appears to have some of the characteristics of an ideal criterion. It is reliable, in that it selects a damping ratio of about 0.7, a value which is commonly considered to be optimum. It is fairly easy to apply, given the time responses of a system. It is also selective, in that the difference between the optimum value and other values is easy to distinguish. It gives an exaggerated picture, however, of the difference between the "goodness" of a system with a damping ratio slightly less than the "optimum" and one with a damping ratio slightly greater than "optimum".

Several other criteria have been used for evaluating the transient performance of zero displacement error transfer systems which are subjected to input step functions. Oldenbourg and Sartorius[5] and Nims[21] have suggested a criterion based on the minimization of the integral

$$I_1 = \int_0^\infty e\, dt \tag{5}$$

This criterion, called the control area, appears to be satisfactory for system responses which do not overshoot. For systems which have a characteristically underdamped response, however, the control area gives an erroneous indication of merit, since overshoots decrease rather than increase the value of the integral.

Curve A of Figure 7 shows the value of the criterion I_1 as a function of the parameter γ, for a system with the transfer function of equation 4 subjected to a unit step function of input. It can be seen that the minimum value of I_1 occurs at the damping ratio $\gamma = 0$. Such a system could hardly be described as optimum. The failure of this criterion to select a reasonable linear second order system is sufficient grounds for its rejection from further consideration.

A modification of this criterion to provide for weighting of the error with time was proposed by Nims. This modified criterion is defined by the integral

$$I_2 = \int_0^\infty t e\, dt \tag{6}$$

The weighted control area, I_2, provides an increasingly heavy penalty for a sustained error, and, as before, that system is considered optimum which produces a minimum value of I_2. Curve B of Figure 7 is a plot of the value of this criterion applied to the second order system. It can be noted that the minimum value of the weighted control area occurs at the damping ratio $\gamma = 0$. Therefore, this criterion fails in the same way as the control area.

Hall[22] has suggested the integral

$$I_3 = \int_0^\infty e^2\, dt \tag{7}$$

as a figure of merit. In this case either positive errors or negative errors due to overshoots will produce positive contributions to the value of the integral. If this criterion is applied to the step function responses of the simple linear

second order transfer system, Curve C of Figure 9 is the result. The minimum criterion value occurs at $\gamma = 0.5$, which is not an unreasonable damping ratio. Furthermore, the criterion can be handled analytically or mechanized with relative ease on a differential analyzer. It also exhibits limited selectivity.

Another figure of merit which has been investigated is given by the integral

$$I_4 = \int_0^\infty |e| \, dt \tag{8}$$

Curve D of Figure 9 shows the results of testing this criterion in connection with the transient response of the simple linear second order transfer system. The minimum value of the criterion occurs at about $\gamma = 0.7$. It is moderately selective, and although it is not analytic, it is easily mechanized on an analog computer.

If time weighting is introduced, this criterion is modified to

$$I_5 = \int_0^\infty t |e| \, dt \tag{9}$$

This function is known as the <u>integral</u> of <u>time-multiplied absolute-value of error</u> (itae) criterion. If, as before, it is tested on the simple second-order system, curve E of Figure 7 is the result. The minimum occurs at $\gamma = 0.7$. The itae criterion is selective and easy to mechanize on an analog computer. Applications to other systems to test its reliability will be discussed below.

Still other figures of merit can be formed with more complex combinations of error and time weighting. Three such criteria are

$$I_6 = \int_0^\infty t e^2 \, dt \tag{10}$$

$$I_7 = \int_0^\infty t^2 e^2 \, dt \tag{11}$$

$$I_8 = \int_0^\infty t^2 |e| \, dt \tag{12}$$

The values of these criteria as functions of γ for the step function responses of the simple second order system are presented in Figure 8. Although these criteria show promise with respect to reliability and selectivity, they are excluded from further consideration because they are difficult to handle either analytically or on the analog computer.

Of the several criteria mentioned above, only those defined by equations (7), (8), and (9) are considered worthy of further investigation. In order to test the general applicability of these three criteria, they are applied to a second order linear zero-velocity-error system, which has the normalized transfer function

$$\frac{C(s)}{R(s)} = \frac{2\gamma s + 1}{s^2 + 2\gamma s + 1} \tag{13}$$

This transfer function describes a second order servomechanism in which all of the damping is mathematically pure error rate damping. The responses of such

a system to a unit step function of input displacement are shown in Figure 9 for various values of the damping ratio ζ. It is interesting to note that, as the damping ratio is increased above one, the response continues to improve.

The values of the three criteria are plotted as functions of the damping ratio, ζ, in Figure 10. All three criteria indicate improvement in performance as the damping ratio is increased. The itae criterion indicates the greatest over-all selectivity.

As a further test of the general applicability of the criteria defined by equations (7), (8), and (9), they have been applied to linear third order systems characterized by the transfer function

$$\frac{C(s)}{R(s)} = \frac{1}{s^3 + bs^2 + cs + 1} \tag{14}$$

The responses of such a system to a unit step function of input displacement are shown in Figure 11 for various combinations of the parameters b and c. The results of the integral of error squared and integral of absolute value of error criteria as applied to this system are presented in Figure 17. It is clear that these criteria fail in the selection of the optimum system parameters. Additional tests indicate that these criteria become even less selective for higher order systems, and it is doubtful that they are suitable for evaluating the transient performance of general transfer systems.

The itae criterion, on the other hand, retains good selectivity for the third order system, as is evidenced by Figure 13. The minimum value of the criterion occurs for the third order system which has the parameters b = 1.75 and c = 2.15. It is gratifying to note that the step function response of a system with these parameters appears qualitatively to have excellent characteristics of fast rise time and small overshoot.

A number of proposals for non-linear modifications of the basic second order linear servomechanism to improve the transient performance have appeared in the literature.[23,24] The step function responses of three such non-linear systems, together with the corresponding values of the itae criterion, are shown in Figure 14. A decreasing value of the criterion corresponds to a general improvement in the transient response characteristics.

FILTER RESPONSES AND STANDARD FORMS

As noted in the section on synthesis methods, a low pass filter is a duplicator, and other duplicators, including servomechanisms, may be designed by analogy to filters. The so-called "ideal" low pass filter whose frequency response is flat in amplitude to the cut-off frequency and then drops suddenly to zero, and whose phase angle is linear with frequency, has a step function response which is

$$c(t) = \frac{1}{2} + \frac{1}{\pi} \, Si\left[\omega_0\left(t - \frac{\varphi_p}{\omega_0}\right)\right] \tag{15}$$

This function is shown, along with the frequency response function of the "ideal" low pass filter, in Figure 15, which is adapted from reference 2. Wallman has shown, however, that it is not possible to realize physically a filter whose amplitude response drops to zero. The step function response of the "ideal" filter shows a non-physical characteristic in that the response begins before the step function is applied. Nevertheless, the transient is in many respects typical of duplicator responses. Note in particular the delay time and the residual oscillation. In general, transfer systems with good low pass filter characteristics have correspondingly good transient responses.

One of the simplest possible low pass filters which is physically realizable is an n-stage RC-coupled pulse amplifier in which the interstage couplings are identical. Since the transfer function of each individual stage is of the form

$$\frac{E_o(s)}{E_i(s)} = \frac{g_m R}{1 + RCs} \tag{16}$$

the algebra of block diagrams leads to the result that the transfer function of the whole amplifier is, in normalized form

$$\frac{E_o(s)}{E_i(s)} = \frac{K}{(s + 1)^n} \tag{17}$$

The characteristic equation of such a system is the binomial expansion of an appropriate order. The roots of this characteristic equation are identical real negative numbers, and the response of a physical system with such a characteristic equation is composed of equally and critically damped modes. Such a response has been suggested by Imlay[25] as an optimum in connection with the response of an aircraft under the control of an automatic pilot, and by Oldenbourg and Sartorius in connection with regulators and servomechanisms.

The precise definition of desired output afforded by the requirement that all modes of the response be equally and critically damped leads readily and explicitly to the required system transfer function. Since the characteristic function of such a system is the product of equal factors, the coefficients of these characteristic functions are the binomial coefficients. The polynomial characteristic functions with binomial coefficients are tabulated in Table III. They might be considered to be a set of standard forms for the synthesis of duplicators. In fact, lacking more suitable forms, Whiteley has suggested the

binomial coefficients for certain higher order system transfer functions. The step function responses of the binomial filters are shown in Figure 16, and the corresponding frequency response functions in Figure 17. It has been noted that, in the language of operational mathematics, these methods of describing the response are equivalent. Therefore, the adjustment of a linear transfer system's parameters resulting in a transfer function with a unit numerator and a characteristic equation with binomial coefficients guarantees the responses indicated.

The transient responses of the binomial filters are not optimum for many applications in the sense that they are relatively slow. The frequency response characteristics show a corresponding attenuation of even the relatively low frequencies.

Another rather simple configuration for the interstage couplings of a pulse amplifier has been suggested by Butterworth[26]. These interstage circuits are designed so that the poles of the normalized system transfer function are evenly distributed on the unit circle in the left half of the complex plane. Such a location of the poles (roots of the characteristic equation) is illustrated for systems of orders one through eight in Figure 18. The corresponding step function and frequency responses are shown in Figures 19 and 20.

Interestingly enough, Whiteley, in compiling his table of standard forms for unit numerator transfer functions, used the characteristic equations of the Butterworth filters for the first four orders. The frequency response characteristics may be seen to fulfill the condition which he stated as a criterion - approximately unit amplitude ratio to 0.8 of the natural frequency. In fact, the designation "maximally flat" is given to the Butterworth configuration in filter design theory.

The standard forms (transfer function coefficients) for the Butterworth filters are shown through the eighth order in Table IV. It would be possible to extend this table by analysis since the definition of the distribution of the poles implies the definition of the characteristic function. This has not been done, however, due to the computational labor involved.

Note that the standard forms for the Butterworth filters are similar to those for the binomial filters in that the coefficients are symmetrical. This phenomenon is indeed typical of all characteristic equations whose roots lie on the unit circle in the complex plane. (The binomial characteristic equations are special cases whose roots all lie at the minus one point.)

The step function responses of the Butterworth filters are, by comparison with the responses of the binomial filters, faster, and not surprisingly, more oscillatory. Nevertheless, for many purposes, they represent a close approach to intuitive concepts of optimum duplicator responses. They served, in each case, as the starting point for the iterative experimental determination of the "optimum" unit numerator transfer functions by the application of the minimum integral of time-multiplied absolute-value of error (itae) criterion.

When the minimum itae criterion is applied to the determination of the optimum unit numerator transfer functions of various orders, the standard forms of Table V are obtained. The corresponding pole locations, step function responses,

and frequency responses are shown in Figures 21, 22, and 23. With regard to these various mathematical equivalents, it can be seen that the application of this arbitrary criterion has not resulted in the selection of a family of systems with similar and progressive characteristics as the order of the system is increased. This is, in a way, a disappointing result, for it had been hoped that it would be possible to extrapolate the experimental results to the selection of standard forms for systems of still higher orders than it was possible to investigate.

Just as the responses of the optimum systems are not greatly different from those of the Butterworth filters, the standard forms are not greatly different either. The greatest difference is in the fifth order standard form. This corresponds to the greatest difference in the time response. In general, the standard forms defined here have coefficients which are slightly higher for the low orders of the complex variable and slightly lower for the higher orders of the complex variable than the corresponding Butterworth standard forms. No interferences on this basis, however, seem warranted.

The criterion has selected responses which are much faster than those of the binomial filters but which are less oscillatory than those of the Butterworth filters. The "goodness" of the selected responses might be classed as a fortunate phenomenon of engineering science. Presumably, the value of the congruent standard forms is correspondingly high.

In the case of linear transfer systems with unit numerator transfer functions (also called zero-displacement-error systems) the itae criterion may be represented as a surface in a multidimensional space which has the dimensions of the transfer function coefficients. The surface (line), its shape and minimum point have already been illustrated for the second and third order cases in Figures 7 and 13. The multidimensional surface itself cannot be graphically represented for systems of order higher than the third. Instead Figures 24 and 25 show sections through the surface for the fourth and fifth order systems. These sections are obtained by adjusting all but one of the coefficients of the system transfer function to their optimum values. This one coefficient is then varied throughout a range on either side of its optimum value and the corresponding variation in the value of the itae criterion is plotted. In Figures 24 and 25 the curves denoted by the letters "b", "c", etc. are the sections through the multidimensional surface obtained by varying the indicated coefficients in this manner. Several step function responses which result from the non-optimum adjustment of a transfer function constant are illustrated adjacent to the corresponding non-minimum value of the criterion function. It may be seen, from these figures, that the adjustment of the coefficients of the lowest powers of the complex variable in the standard form is the most critical both with respect to the character of the step function response and with respect to the value of the criterion. The same situation obtained throughout the investigation, although the higher ordered cases are not illustrated in this way.

The class of linear duplicators with unit numerator transfer functions, while basic, represents only a small fraction of the possible linear systems. The numerators of possible system transfer functions are almost infinitely variable. By confining attention, however, to duplicators with no steady state displacement error, the normalized transfer function numerator polynomials are limited to those which include a constant term of unit magnitude. This would still leave open for consideration a large number of polynomial numerators, of an order equal to or less than the corresponding denominator, if an arbitrary choice of polynomial coefficients were allowed. In servomechanism design practice, however, there are two limiting cases, the zero-velocity-error and zero-acceleration-error systems.

The steady state error of a servomechanism may be shown to be

$$\underset{t \to \infty}{e(t)} = C_0 r + C_1 \frac{dr}{dt} + \frac{C_2}{2} \frac{d^2 r}{dt^2} + \ldots\ldots \tag{18}$$

In terms of the generalized transfer function constants shown in the transfer function

$$\frac{C(\lambda)}{R(\lambda)} = \frac{P_m \lambda^m + P_{m-1} \lambda^{m-1} + \ldots + P_2 \lambda^2 + P_1 \lambda + P_0}{Q_n \lambda^n + Q_{n-1} \lambda^{n-1} + \ldots + Q_2 \lambda^2 + Q_1 \lambda + Q_0} \tag{19}$$

displacement error coefficient $C_0 = 0$ when $P_0 = Q_0$
velocity error coefficient $C_1 = 0$ when $P_1 = Q_1$
acceleration error coefficient $C_2 = 0$ when $P_2 = Q_2$

Therefore, in considering the zero-velocity and zero-acceleration-error systems as limiting cases from among all the possible systems with polynomial numerator transfer functions, only two different numerators for each order of the denominator will be selected. The optimum transfer functions (standard forms) for systems with normalized transfer functions

$$\frac{C(s)}{R(s)} = \frac{1}{s^n + q_{n-1} s^{n-1} + \ldots + q_2 s^2 + q_1 s + 1} \tag{20}$$

have already been found. The possibilities of finding standard forms for zero-velocity-error systems characterized by the transfer function

$$\frac{C(s)}{R(s)} = \frac{q_1 s + 1}{s^n + q_{n-1} s^{n-1} + \ldots + q_2 s^2 + q_1 s + 1} \tag{21}$$

and zero-acceleration-error systems with the transfer function

$$\frac{C(s)}{R(s)} = \frac{q_2 s^2 + q_1 s + 1}{s^n + q_{n-1} s^{n-1} + \ldots + q_2 s^2 + q_1 s + 1} \tag{22}$$

will now be examined in turn.

It may be noted that $C_1 = \int_0^\infty e\, dt$ and $C_2 = \int_0^\infty te\, dt$ for the error response to a unit step input, and that low (or zero) values of these integrals have already been rejected as suitable criteria for optimum response. On an experimental basis, at least, no suitable combination of system parameters could be found which would give displacement step function responses in accord with intuitive concepts of a good response for many of the zero-velocity and zero-acceleration-error systems. This is by no means to say, however, that the itae criterion may not be applied to select the "best" possible response.

Figures 9 and 10 have already been presented to show the possible responses of second order zero-velocity-error systems and the corresponding values of the itae criterion. An arbitrary selection of the damping parameter $\gamma = 1.6$, as optimum, may be made on the basis that further increases in the damping parameter result in a negligible improvement in the response.

Very much the same situation obtains with regard to the third order zero-velocity-error system with the transfer function

$$\frac{C(s)}{R(s)} = \frac{cs + 1}{s^3 + bs^2 + cs + 1} \tag{23}$$

For any given value of "b" the step function response will improve indefinitely as the parameter "c" is increased. The value b = 1.75 is optimum according to the itae criterion, and the value c = 3.25 may be selected as marking the onset of diminishing returns.

The standard forms corresponding to the minimum value of the itae criterion for the zero-velocity-error systems through the sixth order are shown in Table VI. The corresponding responses to step functions of input displacement are illustrated in Figure 26. Large peak overshoots and rapid accelerations are a concomitant of zero-velocity error. The alternatives suggested by Whiteley, who used peak overshoot as a criterion, tend to have a persistent error which the itae criterion will not tolerate. A compromise which may have some merit in this case is afforded by the zero-velocity-error systems which have transfer function denominators identical to those of the binomial filters (Table III). The step function responses of these systems are presented in Figure 27. Note that while the responses exhibit less overshoot and less rapid accelerations than the optimum responses of Figure 26, they are, at the same time, appreciably slower.

The case of the third order zero-acceleration-error transfer system is similar to the third order zero-velocity-error system in that the itae criterion diminishes in value indefinitely as the value of the "b" parameter is increased. The value c = 4.9 is optimum and the value b = 3.0 marks the point where little improvement in response results from further increases in this parameter.

Other standard forms for the zero-acceleration-error systems through the sixth order are shown in Table VII, and the corresponding step function responses appear in Figure 28. As in the case of the zero-velocity-error systems the step function responses, while "best" according to the itae criterion, may still leave something to be desired with respect to peak overshoot. The systems defined by Whiteley's standard forms suffer from the same defect as before; that is, they tend to have a persistent error. The binomial zero-acceleration-error systems,

therefore, may again prove to be a suitable compromise. The step function responses of the binomial zero-acceleration-error systems are shown in Figure 29.

It is worth reiterating that while these responses are relatively slow, they may, theoretically at least, be reproduced to an arbitrary time scale. The itae criterion lays heavy emphasis on a rapid response in non-dimensional time, and the overshoots, undershoots, and rapid accelerations are an inevitable feature if rapid responses of zero-acceleration-error systems are required. Another criterion, such as $\int_0^\infty |e|\, dt$, would penalize the first overshoot more heavily, but it is dubious that this criterion would select a zero-acceleration-error system response clearly superior to the one selected by the itae criterion ($\int_0^\infty t\,|e|\,dt$).

None of the higher ordered zero-velocity or zero-acceleration-error systems have good responses as adjudged by intuition. It may be that the procedure of optimizing the response to a step function of input displacement for zero-velocity and zero-acceleration-error systems is misleading. Optimizing the responses to velocity and acceleration inputs, however, would lead to not greatly different standard forms, and the responses to step functions of input displacement would still be relatively poor. Design compromises are indicated. Systems with good responses to step functions of input displacement and small but finite velocity and acceleration errors probably represent an over-all optimum. Standard forms for such systems remain to be discovered.

SUGGESTED APPLICATION OF THE CRITERION AND OF STANDARD FORMS TO DESIGN

If a standard form is available for the type of transfer function involved in a particular design problem, its use represents a simple, powerful and accurate synthesis procedure. The elementary steps involved would be:

1. Write the differential equations of the system.

2. Develop the system transfer function, leaving in literal form the constants which may be adjusted by design.

3. Normalize the transfer function (equivalent to a time scale change in the time domain).

4. Solve algebraically for the values of the design variables which will make the transfer function denominator conform numerically to the appropriate standard form.

5. If it is a matter of choice, the real time scale of the response may be adjusted by the selection of suitable design variables.

6. The system will have the desired response.

Mathematical operations involved comprise the simplest algebra and the direct LaPlace transformation (usually accomplished by inspection). The use of standard forms does not involve solving for the roots of equations, plots or graphical constructions, integration, or inverse LaPlace transformations. It is a true synthesis method in that it leads directly and unequivocally to a description of the required system in terms of its design parameters.

The standard forms for the zero-displacement-error systems appear to have immediate application to the design of the many servomechanisms, regulators, and instruments which have transfer functions with unit numerators. They further give rise to a new class of multistage pulse amplifiers with a novel and unique adjustment for optimum step function response.

In those cases where standard forms are unavailable for the exact type of transfer function involved, the use of the most nearly approximate standard form will lead to a very rapid estimate of suitable system adjustments. This estimate may then be refined by other methods. Of course, it may be hoped that standard forms will eventually be developed for all cases of practical interest to the designer of linear systems.

Initially, in the experimental synthesis of dynamic systems with parasitic non-linearities, an approximate linear mathematical synthesis is often carried out as a guide. The rapidity and ease with which this might be done by standard forms would seem to recommend them for this purpose.

If analog computation is employed in the study of linear or non-linear systems for which no standard forms are available, the itae criterion may still be used as a unitary figure of merit for the rapid evaluation of a large number of configurations.

In the case of systems with multiple outputs, such as an aircraft under automatic control, the criterion, suitably weighted, could be applied to the several outputs simultaneously and the sum of the weighted itae criterion values would be an over-all figure of merit for the system.

Finally, since the application of the criterion requires an origin in time, it may appear that the standard forms developed from it are not suitable for the optimum synthesis of servomechanisms operating on continuous signals in the presence of statistical noise. The fundamental concept of linear filter theory, however, is discrimination against the noise on a frequency basis. Phillips, as has been pointed out above, suggests the selection of a _form_ for the servomechanism system transfer function and subsequent adjustment of a design parameter to minimize the root mean square error in the presence of given spectra of signal and noise. There is no reason why the form which is selected should not be one of the standard forms defined by application of the itae criterion. The design parameter which is reserved for the optimizing process is the time scale of the standard form response. This is the equivalent of saying that the natural frequency of the system is placed between the signal frequencies and the noise frequencies in a way which gives the least root mean square error in following the signal and rejecting the noise.

CONCLUSION

Standard forms can provide a quick and easy method for the synthesis of optimum dynamic response in a variety of applications. Where, as in very high ordered linear, non-linear, or multiple output systems, the available standard forms themselves are not applicable, the itae criterion still has exceptional merit of its own, and permits the rapid and unequivocal experimental selection of optimum system adjustments. There does not, however, appear to be any theoretical limit to the number of standard forms which may be developed for both general and special applications to linear systems. Eventually a table of standard forms similar to a complete table of LaPlace transforms should be available for all cases of interest. At that time the synthesis of linear systems to have optimum transient response will become a simple, straightforward matter of algebra instead of the involved and often baffling problem which it has been in the past.

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APPENDIX I. DEFINITION OF SYMBOLS

$r(t)$	=	input to a transfer system
$c(t)$	=	output, or response, of a transfer system
$e(t)$	=	$r(t) - c(t)$ = transfer system error
$E(s)$	=	$R(s) - C(s)$ = LaPlace transform of error
$w(t)$	=	transfer system weighting function
$W(s)$	=	$C(s)/R(s)$ = transfer function of a linear system
$W(j\omega)$	=	frequency response function
j	=	$\sqrt{-1}$
ω	=	angular frequency
ω_0	=	natural angular frequency
λ	=	LaPlace complex variable
s	=	λ/ω_0 = normalized LaPlace variable
t	=	time
P_i, Q_i	=	transfer function coefficients
p_i, q_i	=	normalized transfer function coefficients
$a_i, b, c, d, \ldots$	=	normalized transfer function coefficients
ζ	=	damping ratio
$\alpha + j\beta$	=	a root of a transfer system characteristic equation
φ_0	=	phase angle at the frequency ω_0
C_i	=	error coefficient
$Si(x)$	=	sine integral function

APPENDIX II. DERIVATION OF THE NORMALIZED TRANSFER FUNCTION

In general, linear transfer systems of the class known as <u>duplicators</u> may be described by an explicit differential equation of the form

$$\left(Q_n \frac{d^n}{dt^n} + Q_{n-1}\frac{d^{n-1}}{dt^{n-1}} + \ldots + Q_2\frac{d^2}{dt^2} + Q_1\frac{d}{dt} + Q_0\right)c(t)$$
$$= \left(P_m \frac{d^m}{dt^m} + \ldots + P_2\frac{d^2}{dt^2} + P_1\frac{d}{dt} + P_0\right)r(t) \tag{24}$$

in which P_i and Q_i are constants and $m \leqq n$. The corresponding transfer function is

$$\frac{C(\lambda)}{R(\lambda)} = \frac{P_m\lambda^m + \ldots + P_2\lambda^2 + P_1\lambda + P_0}{Q_n\lambda^n + \ldots + Q_2\lambda^2 + Q_1\lambda + Q_0} \tag{25}$$

Equation 25 may be put in a more convenient form for some purposes by a normalized process, which is accomplished as follows:

1. Define a constant ω_0 such that

$$\omega_0^n = \frac{Q_0}{Q_n} \tag{26}$$

2. Define new coefficients for the denominator terms in equation 25 by

$$q_i = \frac{Q_i}{\omega_0^{n-i}Q_n} \qquad i = 1, 2, \ldots n \tag{27}$$

and new coefficients for the numerator terms by

$$p_i = \frac{P_i}{\omega_0^{n-i}Q_n} \qquad i = 0, 1, 2 \ldots m \tag{28}$$

3. Divide the numerator and denominator of equation 25 by Q_n and apply the definitions of 26, 27, and 28. The transfer function then becomes

$$\frac{C(\lambda)}{R(\lambda)} = \frac{p_m\omega_0^{n-m}\lambda^m + \ldots + p_1\omega_0^{n+1}\lambda + p_0\omega_0^n}{\lambda^n + q_{n-1}\omega_0\lambda^{n-1} + \ldots + q_1\omega_0^{n+1}\lambda + \omega_0^n} \tag{29}$$

4. Introduce a new complex variable $\underline{s}$ such that

$$s = \frac{\lambda}{\omega_0} \tag{30}$$

Then the transfer function reduces finally to the normalized form

$$\frac{C(s)}{R(s)} = \frac{P_m s^m + \ldots + P_1 s + P_0}{s^n + q_{n-1} s^{n-1} + \ldots + q_2 s^2 + q_1 s + 1} \tag{31}$$

Equation 30 is equivalent to the substitution of a new independent variable in the original differential equation, where $T = \omega_0 t$. It is important to note that the transfer function of the system has been reduced to a form in which the coefficients of the first and last terms of the denominator are unity.

APPENDIX III. ANALOG COMPUTER TECHNIQUES

The development of the standard forms outlined in this paper depended on the availability of the transient responses associated with a very large number of combinations of transfer function coefficients. The computational labor necessary to obtain these responses by conventional operational methods would have been prohibitive, especially for the higher-order systems. An electronic analog computer afforded the only practical means for obtaining the transient responses and the corresponding criterion values.

Figure 30 shows the basic computer circuit diagram for obtaining the response of a second-order zero-displacement-error system which has the normalized transfer function

$$\frac{C(s)}{R(s)} = \frac{1}{s^2 + bs + 1} \tag{32}$$

If $r(t)$ is a unit step function applied at $t = 0$, then $R(s) = 1/s$, and the transform of the output response is

$$C(s) = \frac{1}{s(s^2 + bs + 1)} \tag{33}$$

The transform of the error response is given by

$$E(s) = R(s) - C(s)$$

$$= \frac{1}{s} - \frac{1}{s(s^2 + bs + 1)} \tag{34}$$

$$= \frac{s^2 + bs}{s(s^2 + bs + 1)}$$

$$= (s^2 + bs) C(s)$$

It is possible, with the aid of a computer technique developed by Beck[28], to extract the error response simultaneously from the basic computer circuit which is used to generate the output response. The required connections are indicated in Figure 30.

The LaPlace transforms of the output response and error response of a second-order zero-velocity-error system are given by

$$C(s) = \frac{bs + 1}{s(s^2 + bs + 1)} \tag{35}$$

and

$$E(s) = \frac{s^2}{s(s^2 + bs + 1)} \tag{36}$$

for an input step function. Beck's method may be applied to extract these responses from the basic computer circuit of Figure 30, although the required connections are not shown.

In a similar manner, the output and error responses of zero-displacement-, zero-velocity-, and zero-acceleration-error systems of higher order may be developed without using differentiators, only one basic computer circuit being required for each order.

It is possible to use an electronic analog computer to obtain the values of most of the criteria discussed in this paper. Criteria values are produced simultaneously with the associated responses. The computer circuit diagrams for eight different criteria are illustrated in Figure 31.

The multipliers which were used by the authors in this study were of the servo type. The absolute value unit consisted of a high-gain amplifier, a limiter, double-throw relay, and three computer amplifiers, arranged as in the circuit diagram of Figure 32. The high gain amplifier and limiter combine to deliver an output of zero volts when the voltage x is negative, and -25 volts when x is positive. The driver amplifier performs an inversion, and closes the relay whenever x is positive. The routing of the x signal through the relaycontacts and the inverting and summing amplifiers is such that the sign of the x output is always positive.

TABLE I   THE ROUTH-HURWITZ STABILITY CRITERION

Characteristic Equation	Criterion
$s^2+bs+1 = 0$	$b > 0$
$s^3+bs^2+cs+1 = 0$	$bc-1 > 0$
$s^4+bs^3+cs^2+ds+1 = 0$	$bcd-d^2-b^2 > 0$
$s^5+bs^4+cs^3+ds^2+es+1 = 0$	$bcd+b-d^2-b^2e > 0$
$s^6+bs^5+cs^4+ds^3+cs^2+fs+1 = 0$	$(bcg+hf-d^2-b^2e)e+b^2c-bd$ $-bc^2f-f^2+bfe+cdf > 0$
$s^7+bs^6+cs^5+ds^4+es^3+fs^2+gs+1 = 0$	$(bcd+bf-d^2-b^2e)e+b^2cg-bdg$ $-bc^2f-f^2-bc+d+bfe+cdf > 0$
$s^8+bs^7+cs^6+ds^5+es^4+fs^3+gs^2+hs+1 = 0$	$(bcde+b^2cg+bef-bdg-b^2e^2-bc^2f$ $-d^2e-f^2-bch+dh+bfe+cdf)f$ $+b^2cfg+bc^2dh-b^2ceh-bcd^2g$ $-b^3g-b^2cd+b^3e+b^2edg+b^2gh+bd^2$ $-b^2f-bdfg-cd^2h+cdf^2+d^3g$ $+b^2gh-bfdg-bh^2+dfh > 0$

TABLE II   METHODS OF SERVOMECHANISM SYNTHESIS

Method	Author	Assumed Input	Criterion	Remarks
Stability Diagram	Routh-Hurwitz Brown	None necessary	Stability	Determines stability only. Brown shows how the degree of stability may be determined.
Frequency Response	Nyquist Bode Nichols	Constant amplitude sinusoids of all frequencies	Stability, maximum gain, stability margins	Most widely used method. Depends on rules of thumb for shaping the frequency response. Essentially a cut and try method.
Root Locus	Evans Moore	None necessary	Stability and degree of stability. Transient and frequency response may be inferred and suitable criteria applied	Criteria are not explicit. A graphical method easy to apply. Only one gain may be adjusted at a time.
RMS-Error	Wiener Phillips Bubb	Input and noise must be stationary time series. Power spectral density must be known.	Minimum root mean square error.	A powerful method, but difficult to apply. Leads explicitly to required transfer characteristic.
Transient Response	Draper et al. Brown-Hall Oldenbourg- Sartorius Bretoi Others	Step function or other simple type	Speed of response, overshoot, minimum $\int e^2$, error coefficients	Charts available for first, second and third order systems. Higher ordered systems must be cut and tried. Analog computers a great help.
Standard Forms	Whiteley	Step function	Maximally flat frequency response, or a given peak overshoot.	The standard form essentially is the desired transfer characteristic. Very easy to apply. The criteria may be questioned.

Table III

THE BINOMIAL STANDARD FORMS

$$s+\omega_0$$
$$s^2+2\omega_0 s+\omega_0^2$$
$$s^3+3\omega_0 s^2+3\omega_0^2 s+\omega_0^3$$
$$s^4+4\omega_0 s^3+6\omega_0^2 s^2+4\omega_0^3 s+\omega_0^4$$
$$s^5+5\omega_0 s^4+10\omega_0^2 s^3+10\omega_0^3 s^2+5\omega_0^4 s+\omega_0^5$$
$$s^6+6\omega_0 s^5+15\omega_0^2 s^4+20\omega_0^3 s^3+15\omega_0^4 s^2+6\omega_0^5 s+\omega_0^6$$
$$s^7+7\omega_0 s^6+21\omega_0^2 s^5+35\omega_0^3 s^4+35\omega_0^4 s^3+21\omega_0^5 s^2+7\omega_0^6 s+\omega_0^7$$
$$s^8+8\omega_0 s^7+28\omega_0^2 s^6+56\omega_0^3 s^5+70\omega_0^4 s^4+56\omega_0^5 s^3+28\omega_0^6 s^2+8\omega_0^7 s+\omega_0^8$$

Table IV

THE BUTTERWORTH STANDARD FORMS

$$s+\omega_0$$
$$s^2+1.4\omega_0 s+\omega_0^2$$
$$s^3+2.0\omega_0 s^2+2.0\omega_0^2 s+\omega_0^3$$
$$s^4+2.6\omega_0 s^3+3.4\omega_0^2 s^2+2.6\omega_0^3 s+\omega_0^4$$
$$s^5+3.24\omega_0 s^4+5.24\omega_0^2 s^3+5.24\omega_0^3 s^2+3.24\omega_0^4 s+\omega_0^5$$
$$s^6+3.86\omega_0 s^5+7.46\omega_0^2 s^4+9.13\omega_0^3 s^3+7.46\omega_0^4 s^2+3.86\omega_0^5 s+\omega_0^6$$
$$s^7+4.5\omega_0 s^6+10.1\omega_0^2 s^5+14.6\omega_0^3 s^4+14.6\omega_0^4 s^3+10.1\omega_0^5 s^2+4.5\omega_0^6 s+\omega_0^7$$
$$s^8+5.12\omega_0 s^7+13.14\omega_0^2 s^6+21.84\omega_0^3 s^5+25.69\omega_0^4 s^4+21.84\omega_0^5 s^3+13.14\omega_0^6 s^2+5.12\omega_0^7 s+\omega_0^8$$

Table V

THE MINIMUM ITAE STANDARD FORMS
ZERO DISPLACEMENT ERROR SYSTEMS

$$s+\omega_0$$
$$s^2+1.4\omega_0 s+\omega_0^2$$
$$s^3+1.75\omega_0 s^2+2.15\omega_0^2 s+\omega_0^3$$
$$s^4+2.1\omega_0 s^3+3.4\omega_0^2 s^2+2.7\omega_0^3 s+\omega_0^4$$
$$s^5+2.8\omega_0 s^4+5.0\omega_0^2 s^3+5.5\omega_0^3 s^2+3.4\omega_0^4 s+\omega_0^5$$
$$s^6+3.25\omega_0 s^5+6.60\omega_0^2 s^4+8.60\omega_0^3 s^3+7.45\omega_0^4 s^2+3.95\omega_0^5 s+\omega_0^6$$
$$s^7+4.475\omega_0 s^6+10.42\omega_0^2 s^5+15.08\omega_0^3 s^4+15.54\omega_0^4 s^3+10.64\omega_0^5 s^2+4.58\omega_0^6 s+\omega_0^7$$
$$s^8+5.20\omega_0 s^7+12.80\omega_0^2 s^6+21.60\omega_0^3 s^5+25.75\omega_0^4 s^4+22.20\omega_0^5 s^3+13.30\omega_0^6 s^2+5.15\omega_0^7 s+\omega_0^8$$

Table VI

THE MINIMUM ITAE STANDARD FORMS
ZERO VELOCITY ERROR SYSTEMS

$$s^2+3.2\omega_0 s+\omega_0^2$$
$$s^3+1.75\omega_0 s^2+3.25\omega_0^2 s+\omega_0^3$$
$$s^4+2.41\omega_0 s^3+4.93\omega_0^2 s^2+5.14\omega_0^3 s+\omega_0^4$$
$$s^5+2.19\omega_0 s^4+6.50\omega_0^2 s^3+6.30\omega_0^3 s^2+5.24\omega_0^4 s+\omega_0^5$$
$$s^6+6.12\omega_0 s^5+6.71\omega_0^2 s^4+8.58\omega_0^3 s^3+7.07\omega_0^4 s^2+6.76\omega_0^5 s+\omega_0^6$$

Table VII

THE MINIMUM ITAE STANDARD FORMS
ZERO ACCELERATION ERROR SYSTEMS

$$s^3+2.97\omega_0 s^2+4.94\omega_0^2 s+\omega_0^3$$
$$s^4+3.71\omega_0 s^3+7.88\omega_0^2 s^2+5.93\omega_0^3 s+\omega_0^4$$
$$s^5+3.81\omega_0 s^4+9.94\omega_0^2 s^3+13.44\omega_0^3 s^2+7.36\omega_0^4 s+\omega_0^5$$
$$s^6+3.93\omega_0 s^5+11.68\omega_0^2 s^4+18.56\omega_0^3 s^3+19.3\omega_0^4 s^2+8.06\omega_0^5 s+\omega_0^6$$
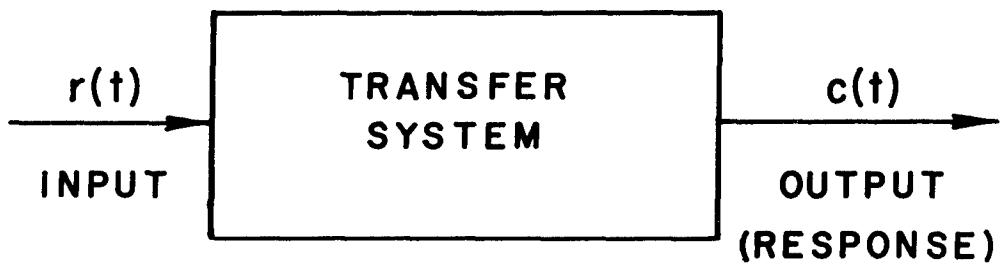
$$C(t) = \int_{-\infty}^{+\infty} r(\tau) \, w(t - \tau) \, d\tau$$

$$C(s) = R(s) W(s)$$

FIGURE I. A TRANSFER SYSTEM
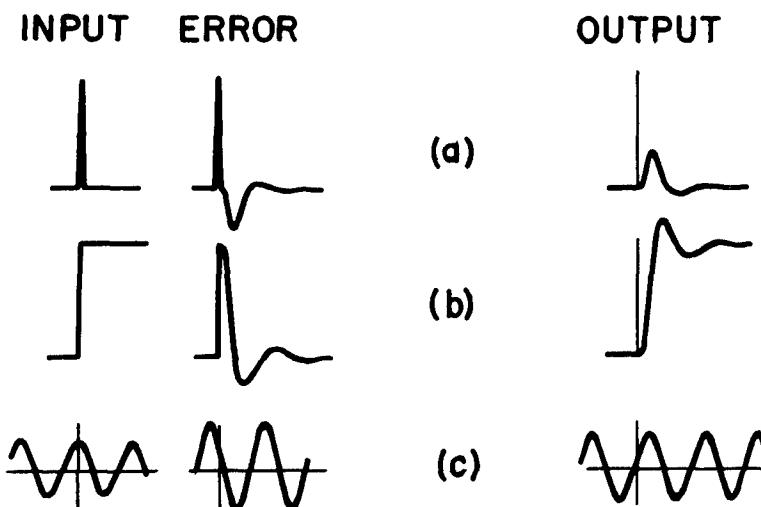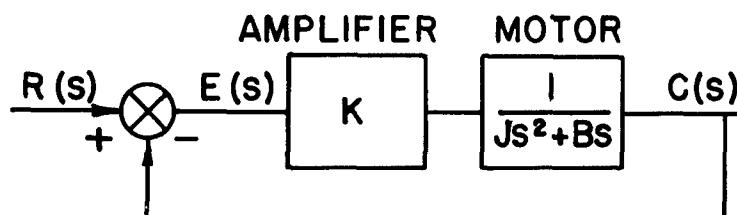
FIGURE 2. A LINEAR POSITIONING SERVO-MECHANISM DEFINED BY THE DIFFERENTIAL EQUATION

$$J\frac{d^2c}{dt^2} + B\frac{dc}{dt} + Kc = Kr$$

AND THE TRANSFER FUNCTION

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Bs + K}$$

TYPICAL RESPONSES ILLUSTRATED ARE:
  (a)   THE IMPULSE RESPONSE, OR WEIGHTING FUNCTION
  (b)   THE STEP FUNCTION RESPONSE
  (c)   THE SINUSOIDAL RESPONSE

α=0 α=.1 α=.2 α=.4 α=.6

α=.8
α=1.0

3 .33

α=1.0

.40

2 β²=.50

α=.8

STABLE

α=.6

β²=.66

COEFFICIENT b

1

α=.4

β²=1.0

α=.2

UNSTABLE

α=.1

β²=1.5

α=0

β²=2.0

β²=2.5

0

0   1   2   3

COEFFICIENT C
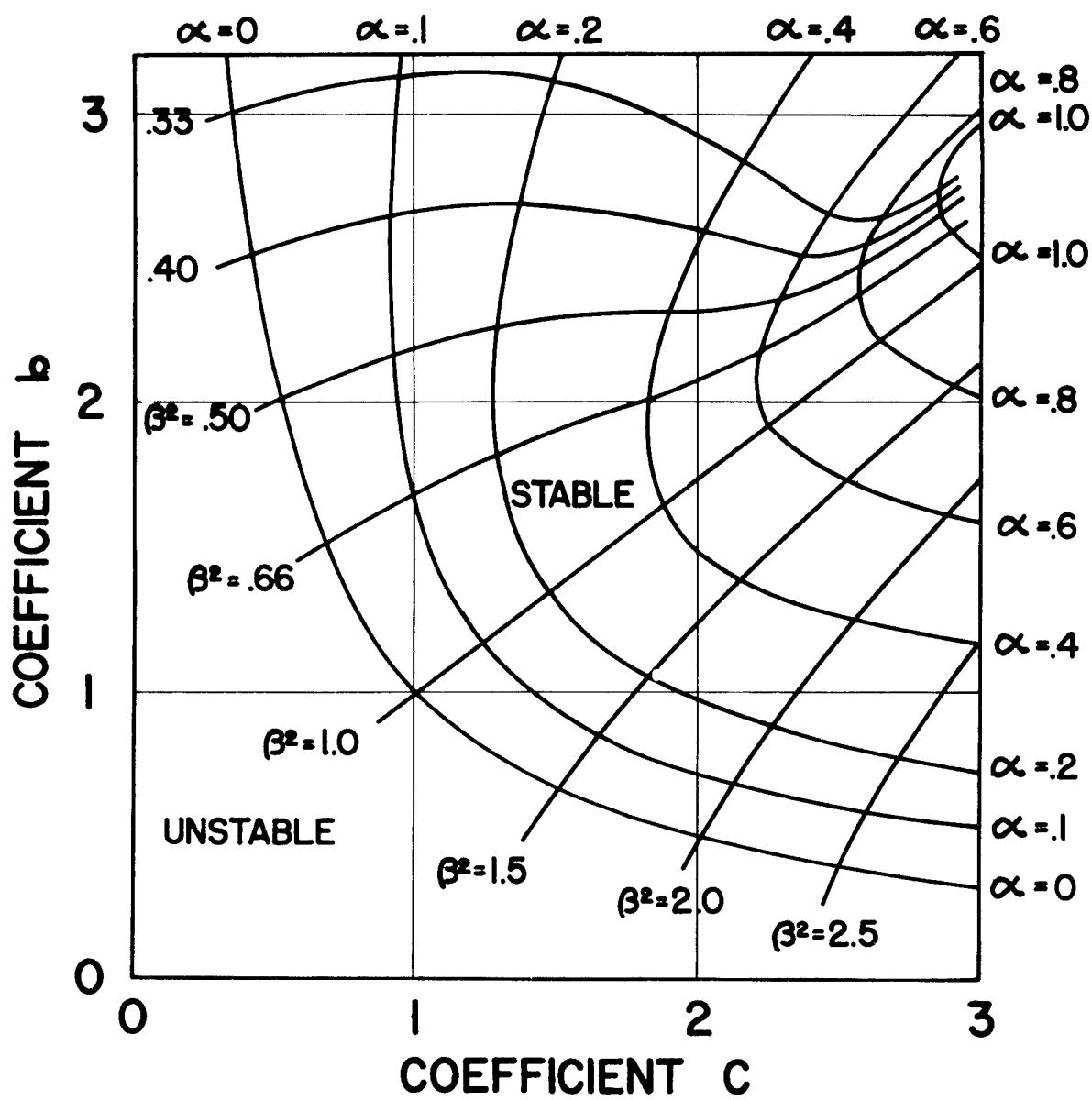
FIGURE 3. STABILITY DIAGRAM FOR THIRD ORDER SYSTEMS DEFINED BY THE TRANSFER FUNCTION

$$\frac{C_{(S)}}{R_{(S)}} = \frac{1}{s^3 + bs^2 + cs + 1} = \frac{1}{(s + \frac{1}{\alpha^2 + \beta^2})\left[(s + \alpha)^2 + \beta^2\right]}$$

SHOWING CONTOURS OF CONSTANT LOGARITHMIC DECREMENT, α, AND UNDAMPENED FREQUENCY SQUARED, β^2.
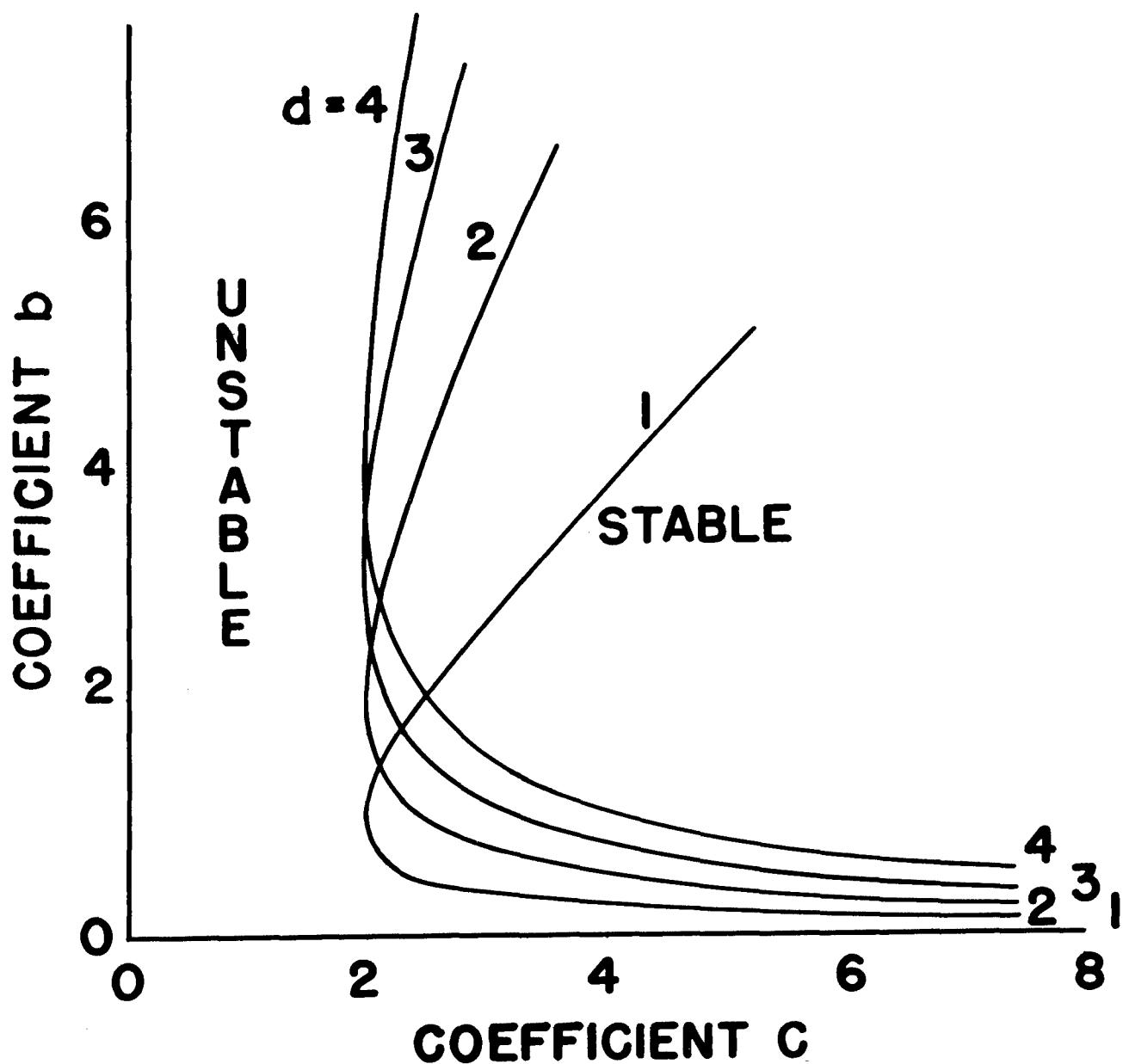
FIGURE 4. STABILITY DIAGRAM FOR FOURTH ORDER SYSTEMS.

FIGURE 6. MEASURES OF THE TRANSIENT PERFORMANCE OF SECOND ORDER SYSTEMS.

FIGURE 5. STEP FUNCTION RESPONSES OF SECOND ORDER SYSTEMS AND FIGURES OF MERIT.

(a) $e(t)$

(b) $\int_o^t e(t)\,dt$

(c) $\int_o^t te\,dt$

(d) $\int_o^t e^2\,dt$

(e) $\int_o^t |e|\,dt$

(f) $\int_o^t t|e|\,dt$
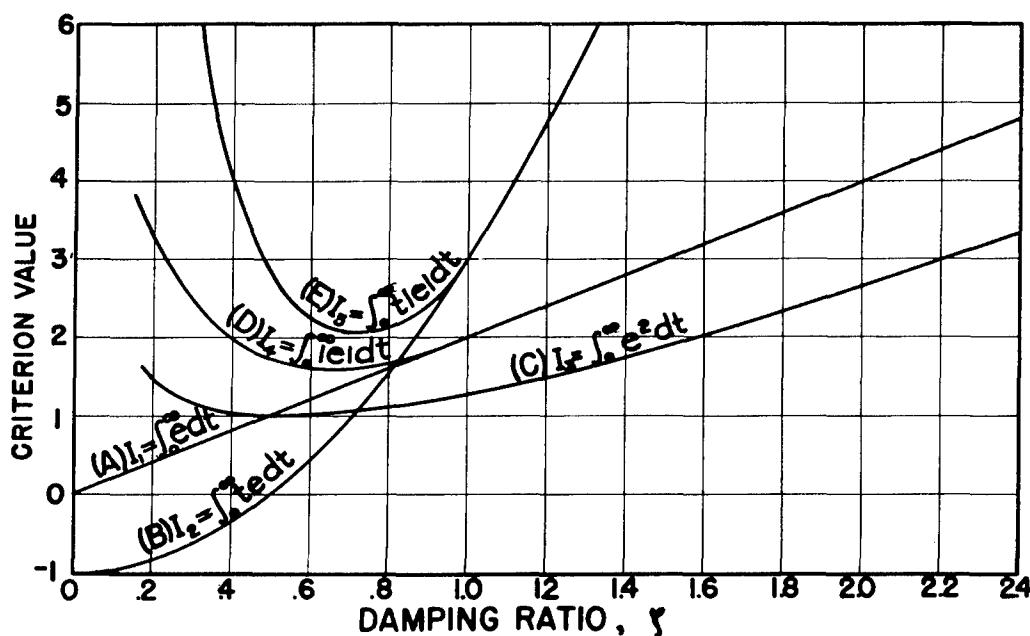
FIGURE 7. CRITERIA FOR THE STEP
FUNCTION RESPONSES OF SECOND
ORDER SYSTEMS

$$\frac{C(s)}{R(s)} = \frac{1}{s^2 + 2\zeta s + 1}$$

CURVE A, INTEGRAL OF ERROR, THE CONTROL AREA.
CURVE B, INTEGRAL OF TIME-MULTIPLIED ERROR,
         THE WEIGHTED CONTROL AREA.
CURVE C, INTEGRAL OF SQUARED ERROR.
CURVE D, INTEGRAL OF ABSOLUTE VALUE OF ERROR.
CURVE E, INTEGRAL OF TIME-MULTIPLIED ABSOLUTE
         VALUE OF ERROR.
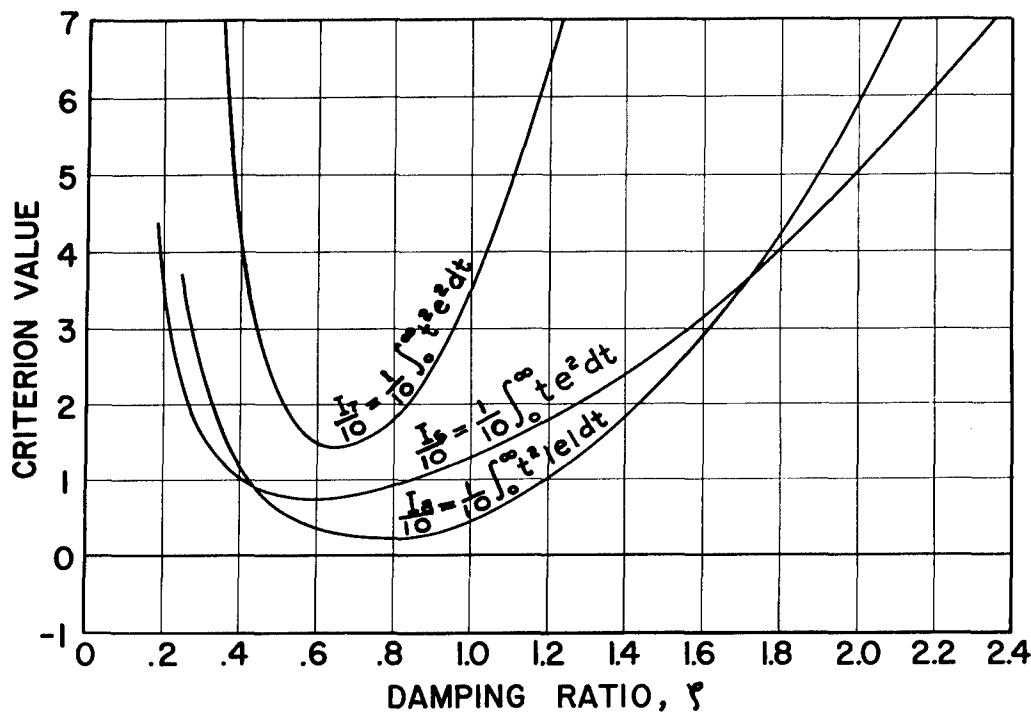
FIGURE 8. THREE ADDITIONAL CRITERIA FOR
THE STEP FUNCTION RESPONSES OF SECOND
ORDER SYSTEMS.

INTEGRAL OF TIME-MULTIPLIED SQUARED ERROR,
INTEGRAL OF TIME-SQUARED ERROR-SQUARED,
AND INTEGRAL OF TIME-SQUARED ABSOLUTE
VALUE OF ERROR.

FIGURE 9. STEP FUNCTION RESPONSES OF ZERO-VELOCITY-ERROR SECOND ORDER SYSTEMS
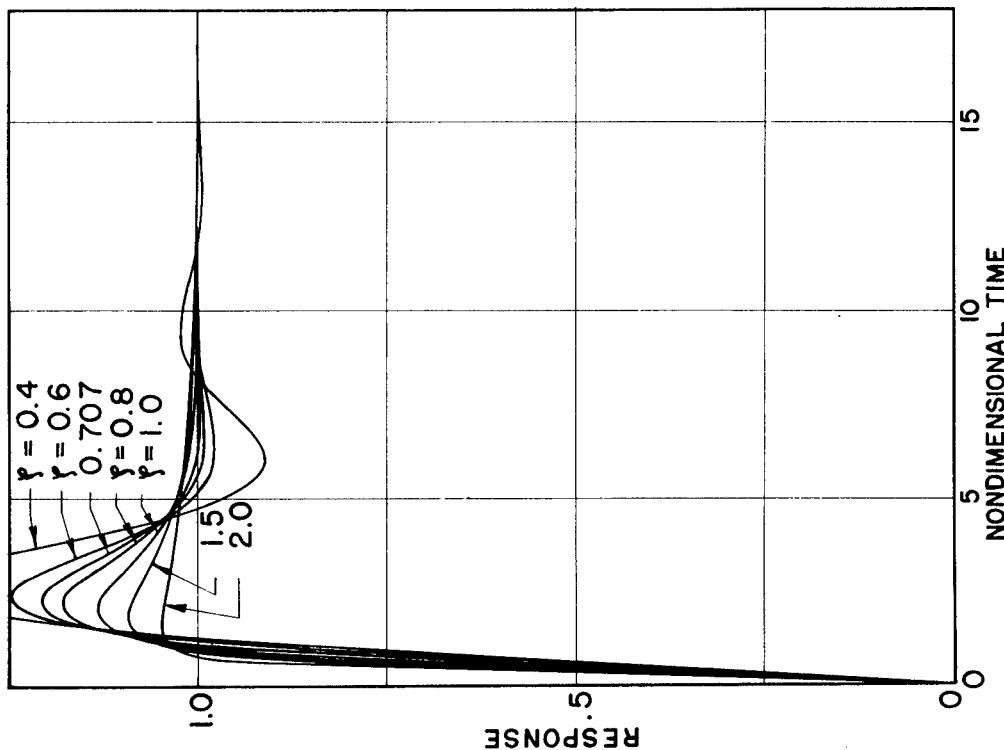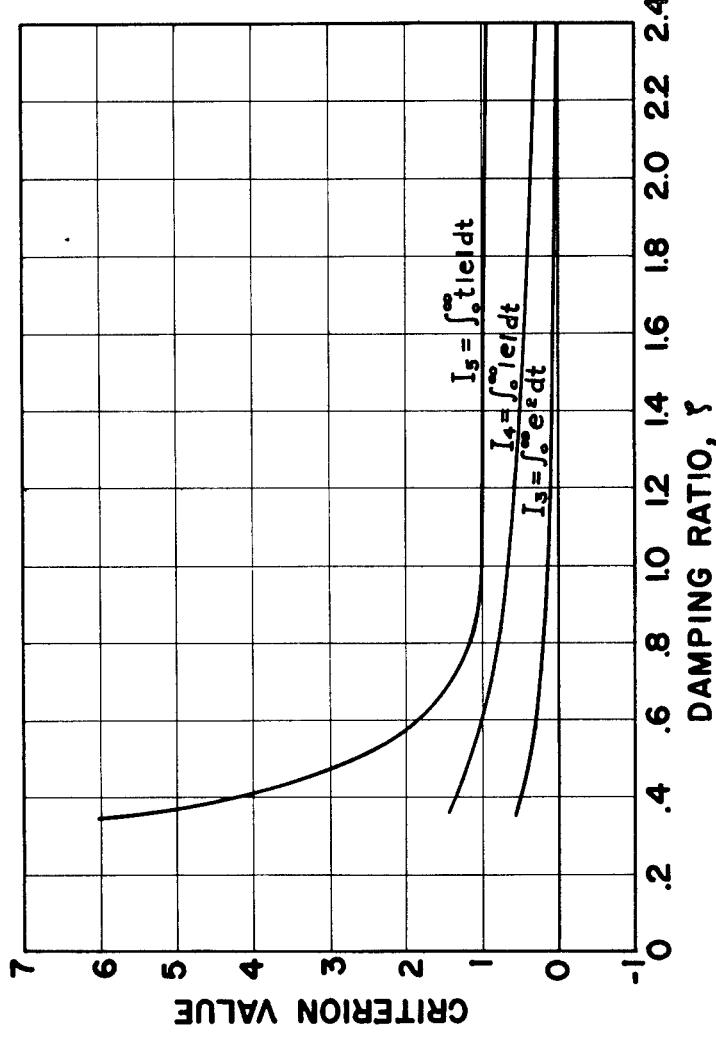
FIGURE 10. CRITERIA APPLIED TO THE STEP FUNCTION RESPONSES OF ZERO-VELOCITY-ERROR SECOND ORDER SYSTEMS.
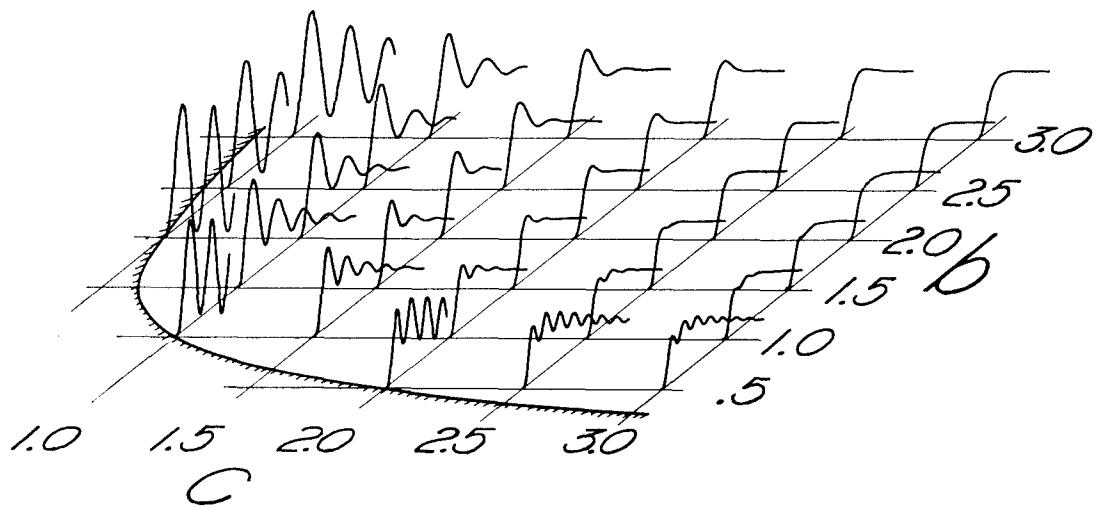
FIGURE II. STEP FUNCTION RESPONSES OF
THIRD ORDER SYSTEMS WITH THE TRANSFER
FUNCTION
$$\frac{C_{(s)}}{R_{(s)}} = \frac{1}{s^3 + bs^2 + cs + 1}$$
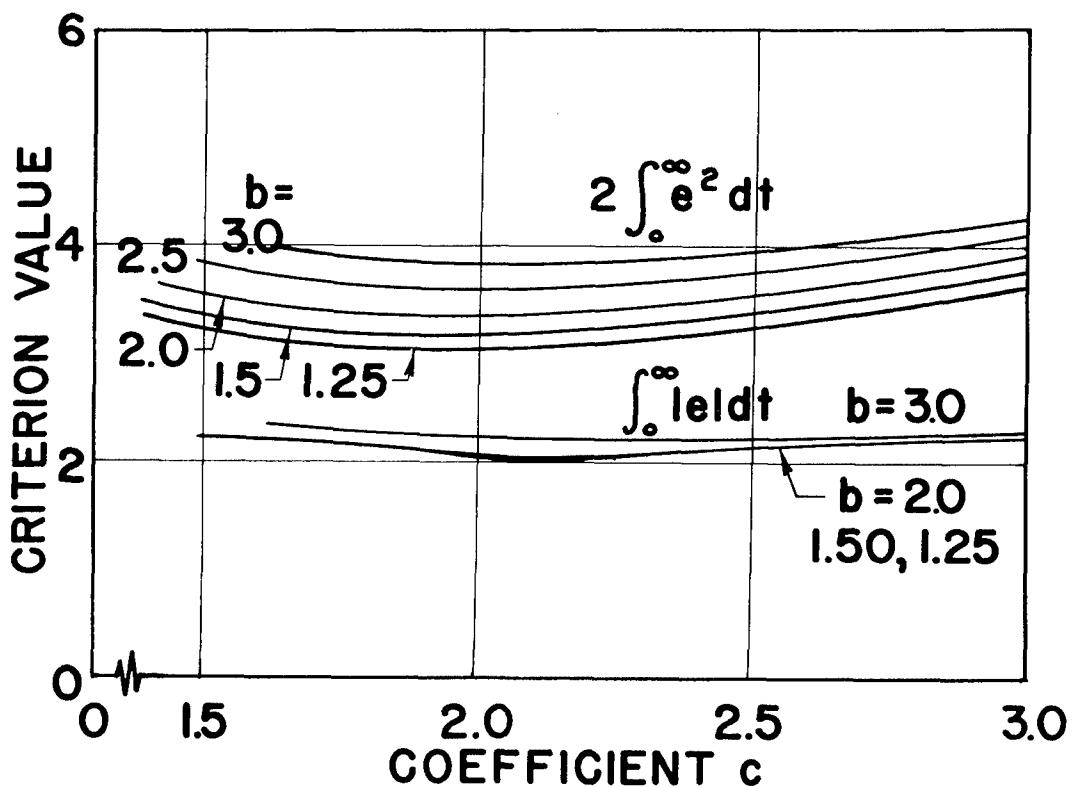
FIGURE 12. INTEGRAL OF SQUARED ERROR
AND INTEGRAL OF ABSOLUTE VALUE OF
ERROR CRITERIA APPLIED TO THE STEP
FUNCTION RESPONSES OF THIRD ORDER
SYSTEMS.
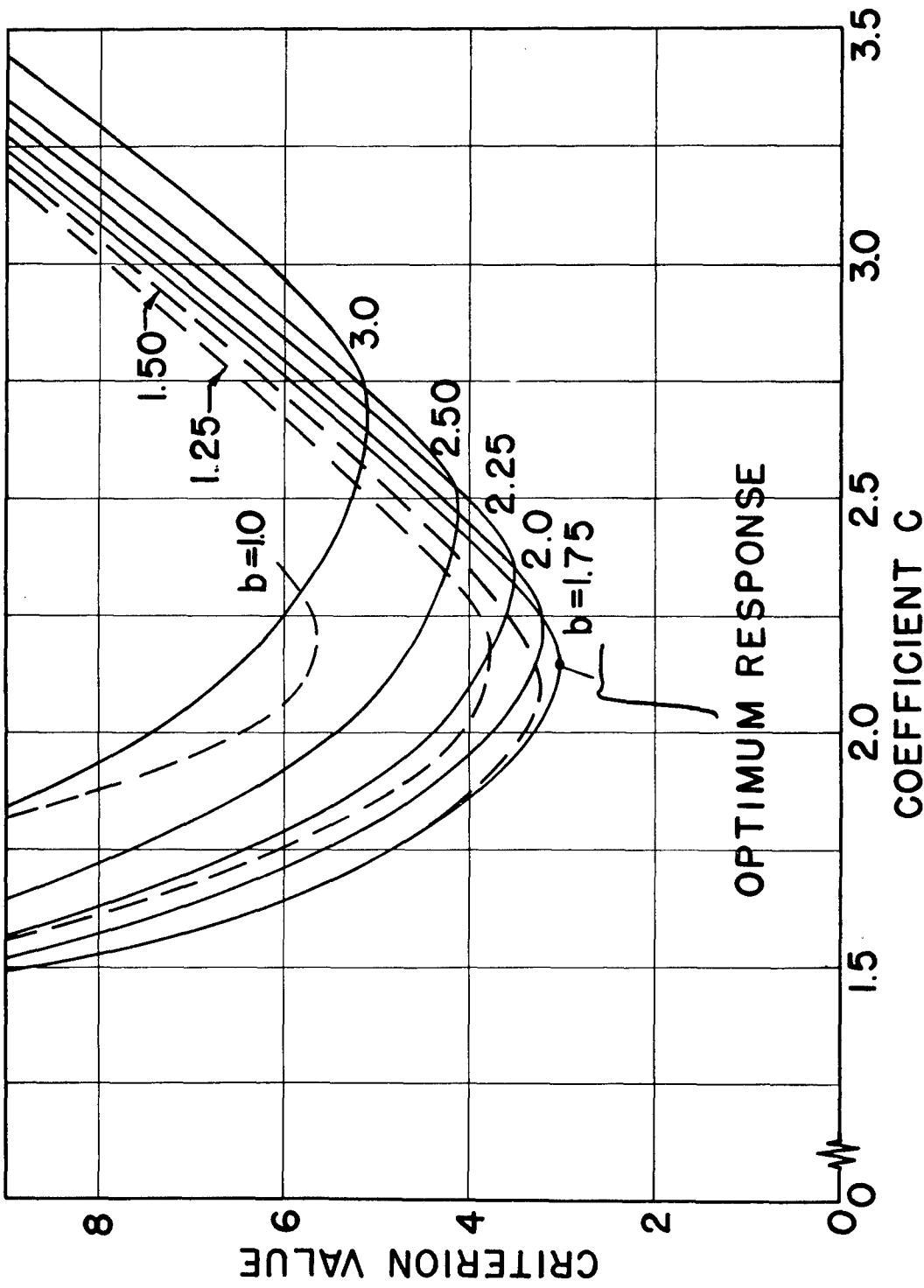
FIGURE 13. THE INTEGRAL OF TIME-MULTIPLIED
ABSOLUTE VALUE OF ERROR CRITERION APPLIED
TO THE STEP FUNCTION RESPONSES OF THIRD
ORDER SYSTEMS.

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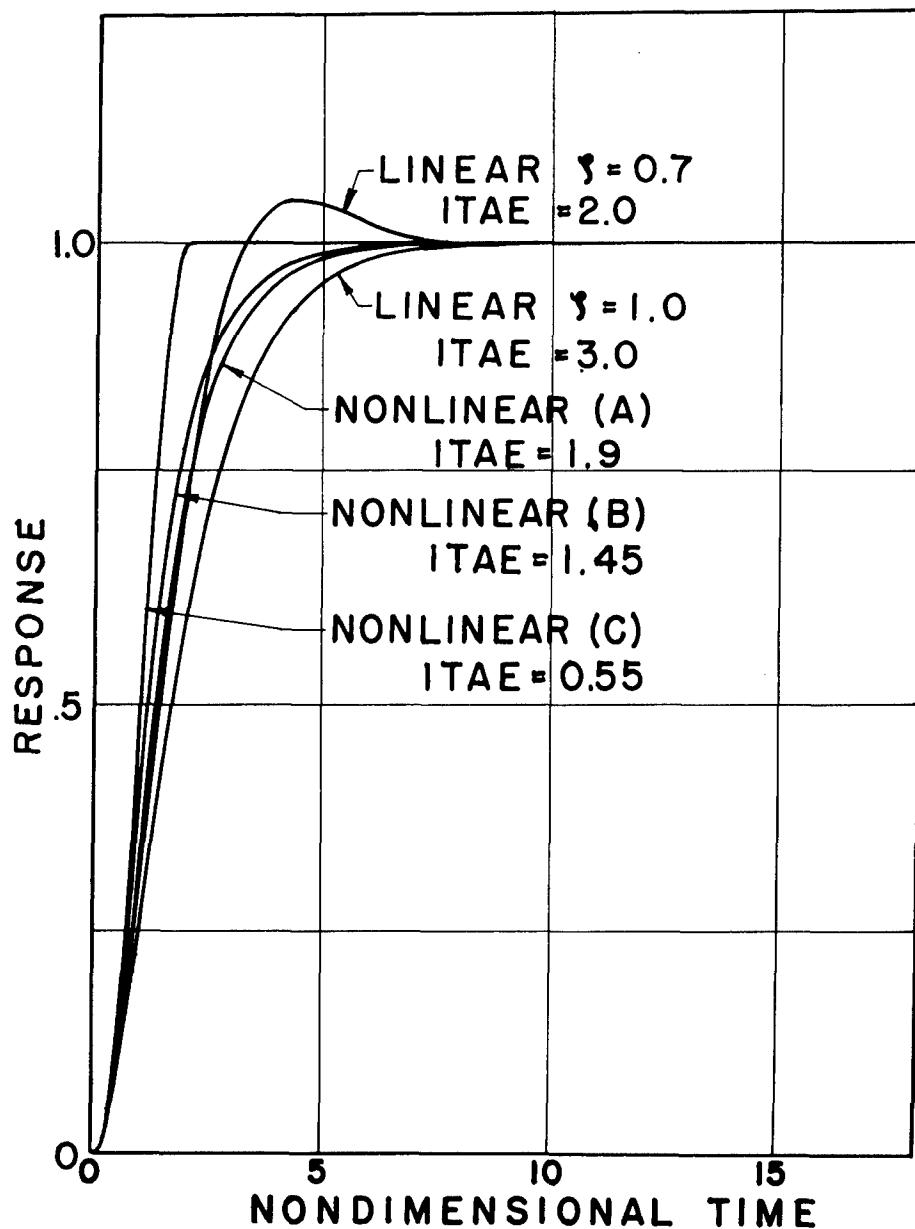
FIGURE 14. STEP FUNCTION RESPONSES
OF LINEAR AND NON-LINEAR SECOND
ORDER SERVOMECHANISMS.

SERVO A $\dfrac{d^2 e}{dt^2} + \dfrac{2}{1+4(e+\frac{de}{dt})}\ \dfrac{de}{dt} + e = 0$

SERVO B $\dfrac{d^2 e}{dt^2} + k\dfrac{de}{dt} + e = 0$ $k=0;\ \ e+\dfrac{de}{dt} \gneq 0$

$k=2;\ \ e+\dfrac{de}{dt} = 0$

SERVO C $\dfrac{d^2 e}{dt^2} + k = 0$ $k=1;\ \ 2e+\dfrac{de}{dt}\left|\dfrac{de}{dt}\right| > 0$

$k=-1;\ \ 2e+\dfrac{de}{dt}\left|\dfrac{de}{dt}\right| < 0$
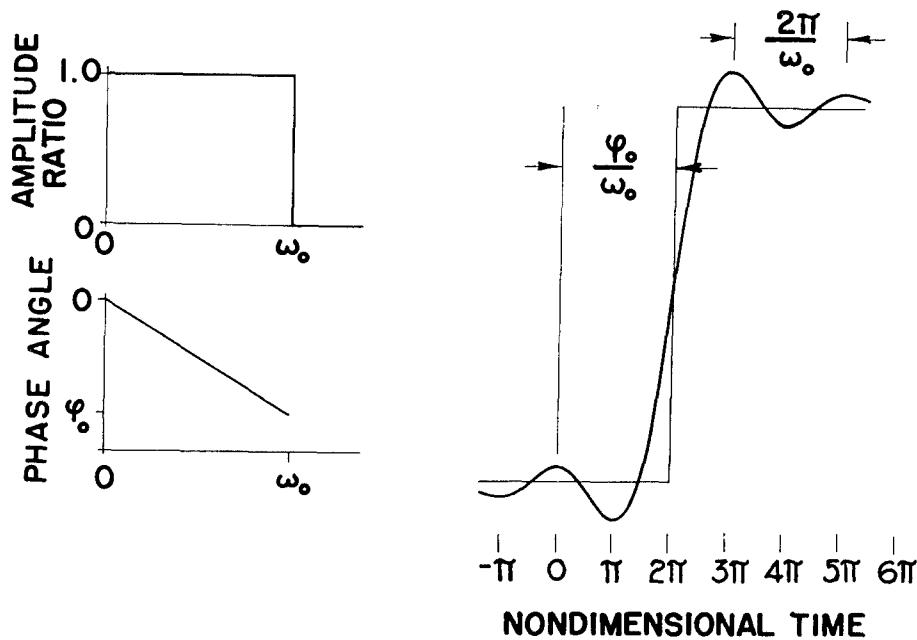
FIGURE 15. STEP FUNCTION RESPONSE
OF THE IDEAL LOW PASS FILTER.
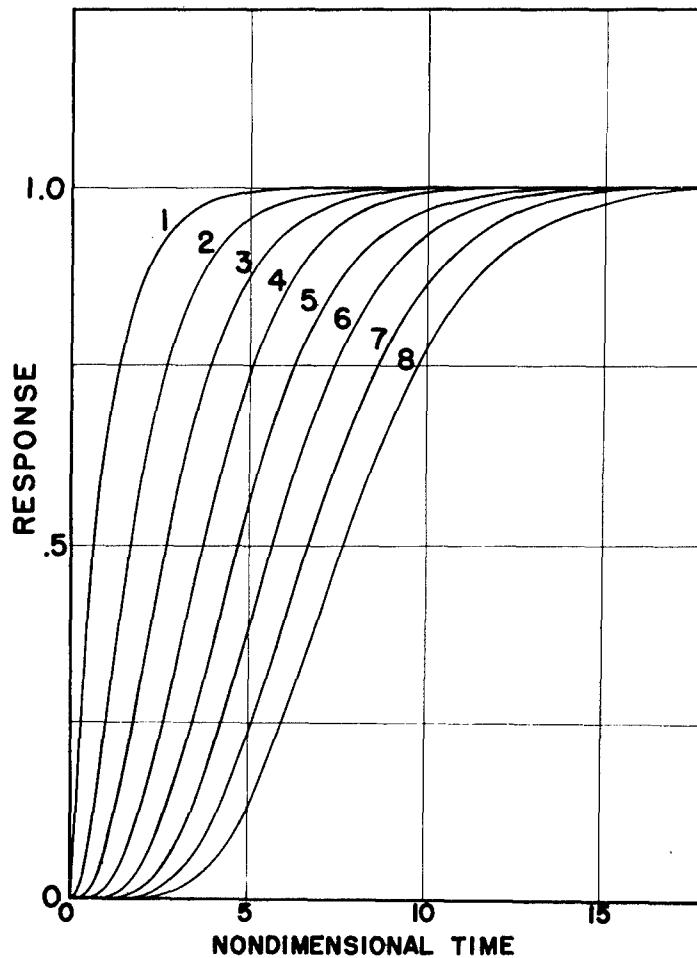
FIGURE 16. STEP FUNCTION RESPONSE
OF "BINOMIAL" FILTERS, DEFINED BY
THE TRANSFER FUNCTIONS

$$\frac{C(s)}{R(s)} = \frac{1}{(s+1)^n} \qquad n = 1, 2, \cdots\cdots 8$$
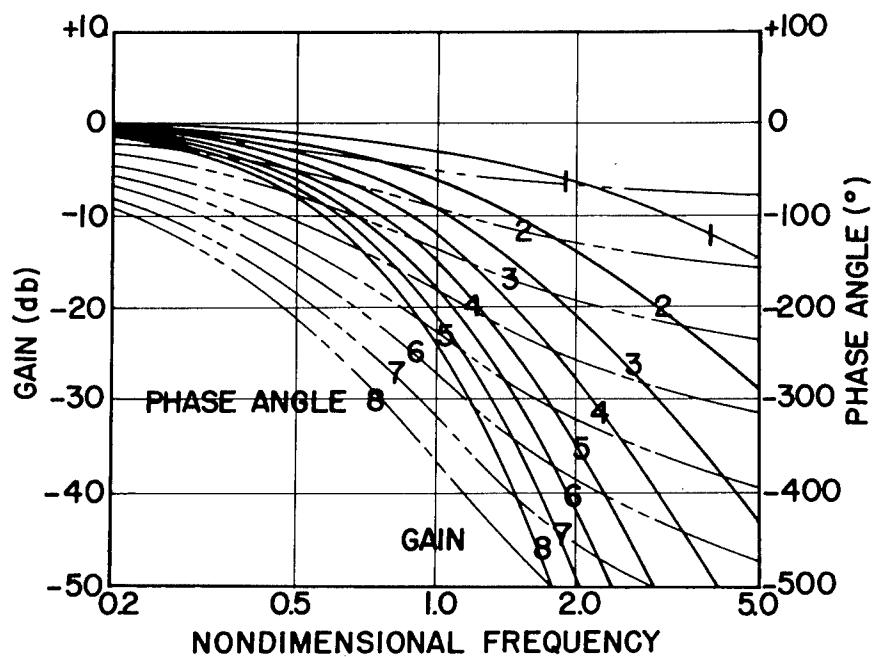
FIGURE 17. FREQUENCY RESPONSE FUNCTIONS OF THE BINOMIAL FILTERS,

$$\frac{C(j\omega)}{R(j\omega)} = \frac{1}{(j\omega+1)^n} \qquad n = 1, 2, \cdots 8$$
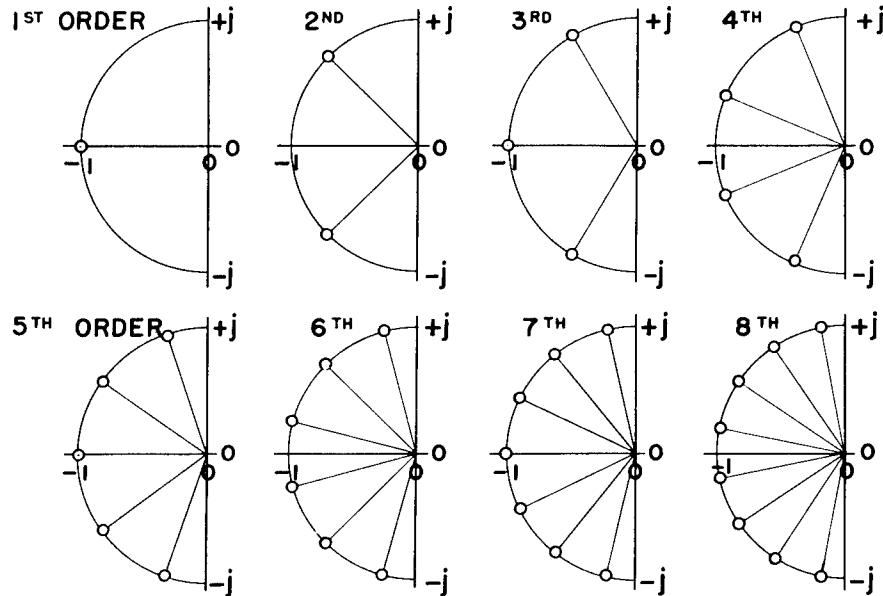
FIGURE 18. POLE LOCATIONS OF THE BUTTER-WORTH FILTERS, FIRST TO EIGHTH ORDERS.
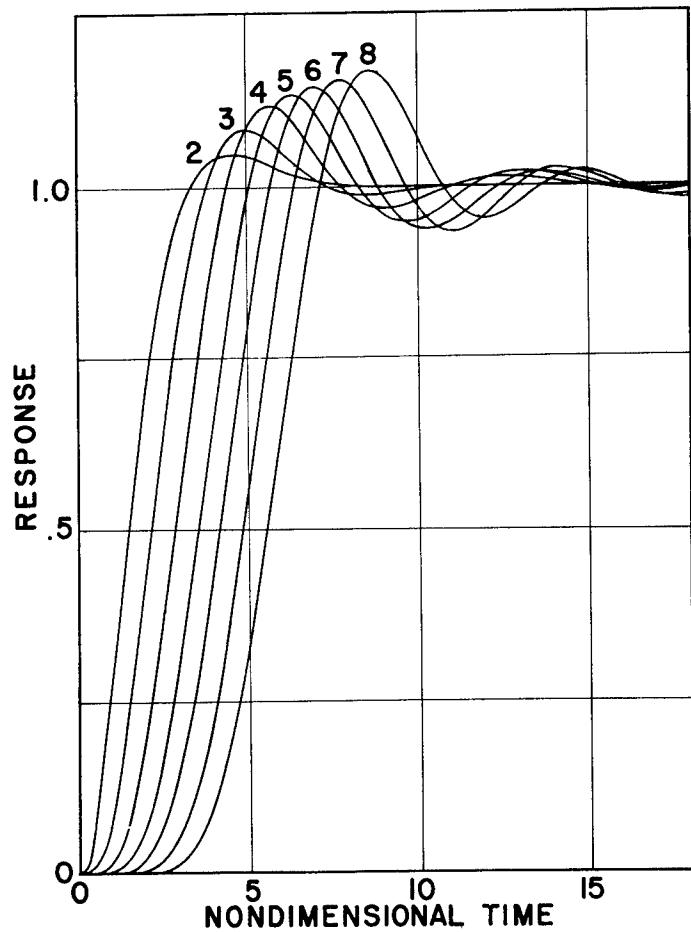
FIGURE 19. STEP FUNCTION RESPONSES OF THE
BUTTERWORTH FILTERS, SECOND TO EIGHTH
ORDERS.
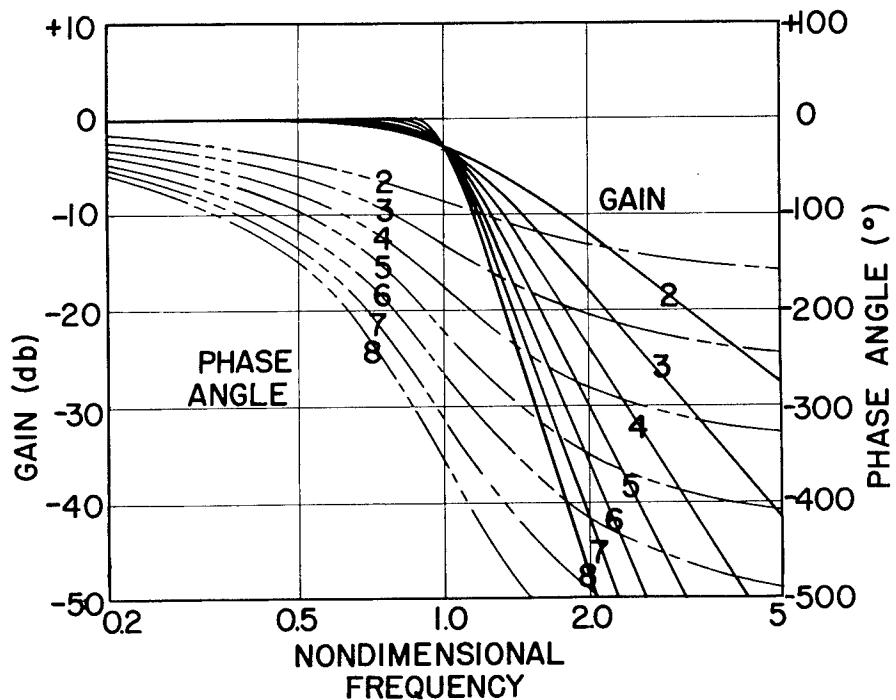
FIGURE 20. FREQUENCY RESPONSE FUNCTIONS
(AMPLITUDE AND PHASE) OF THE BUTTER –
WORTH FILTERS, SECOND TO EIGHTH ORDERS.
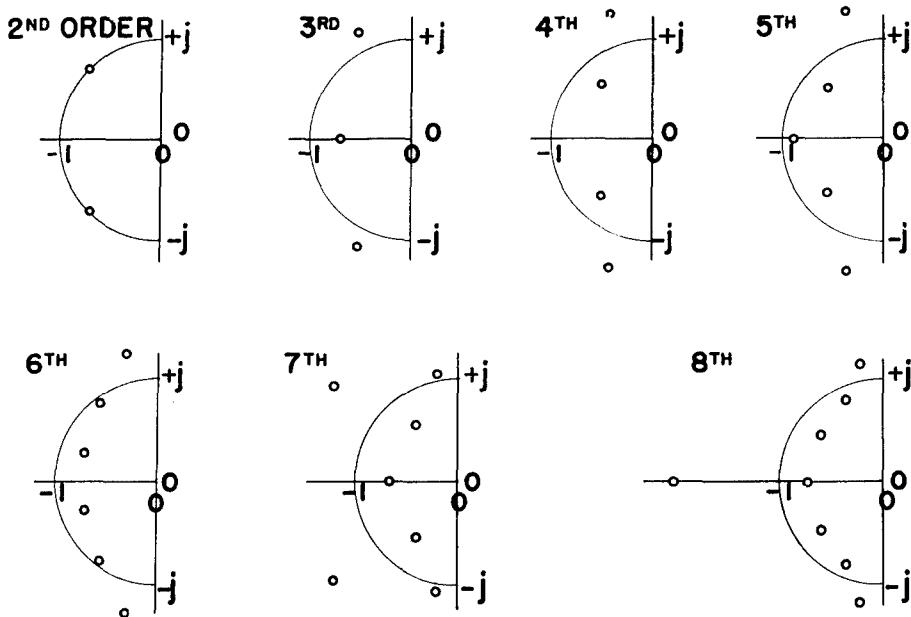
FIGURE 21. POLE LOCATIONS OF THE OPTIMUM
UNIT-NUMERATOR TRANSFER SYSTEMS, SECOND
TO EIGHTH ORDERS.
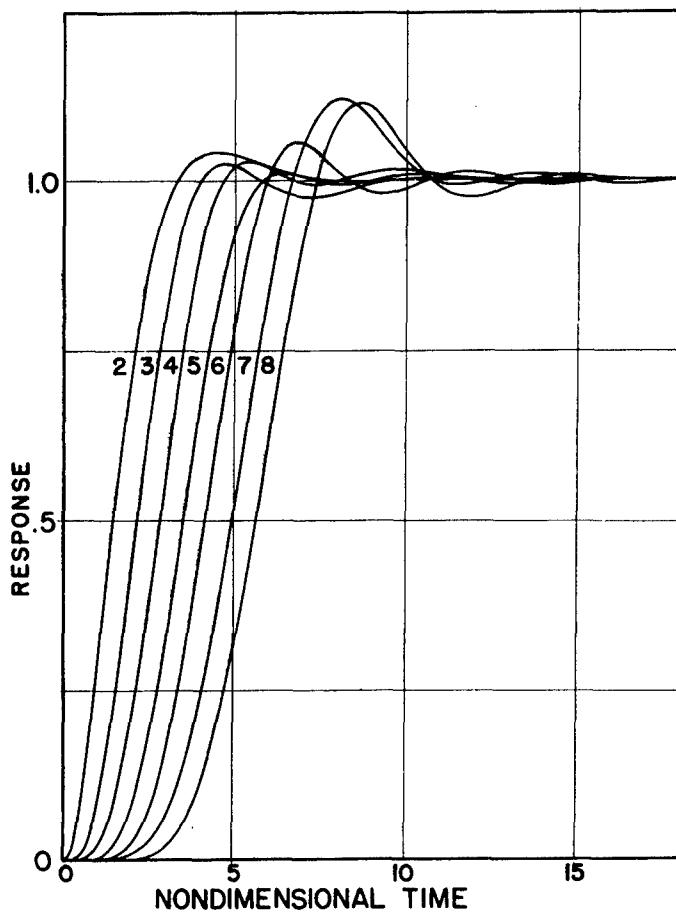
FIGURE 22. STEP FUNCTION RESPONSES OF THE
OPTIMUM UNIT-NUMERATOR TRANSFER SYSTEMS,
SECOND TO EIGHTH ORDERS. THESE RESPONSES
HAVE A MINIMUM INTEGRAL OF TIME-MULTIPLIED
ABSOLUTE VALUE OF ERROR.
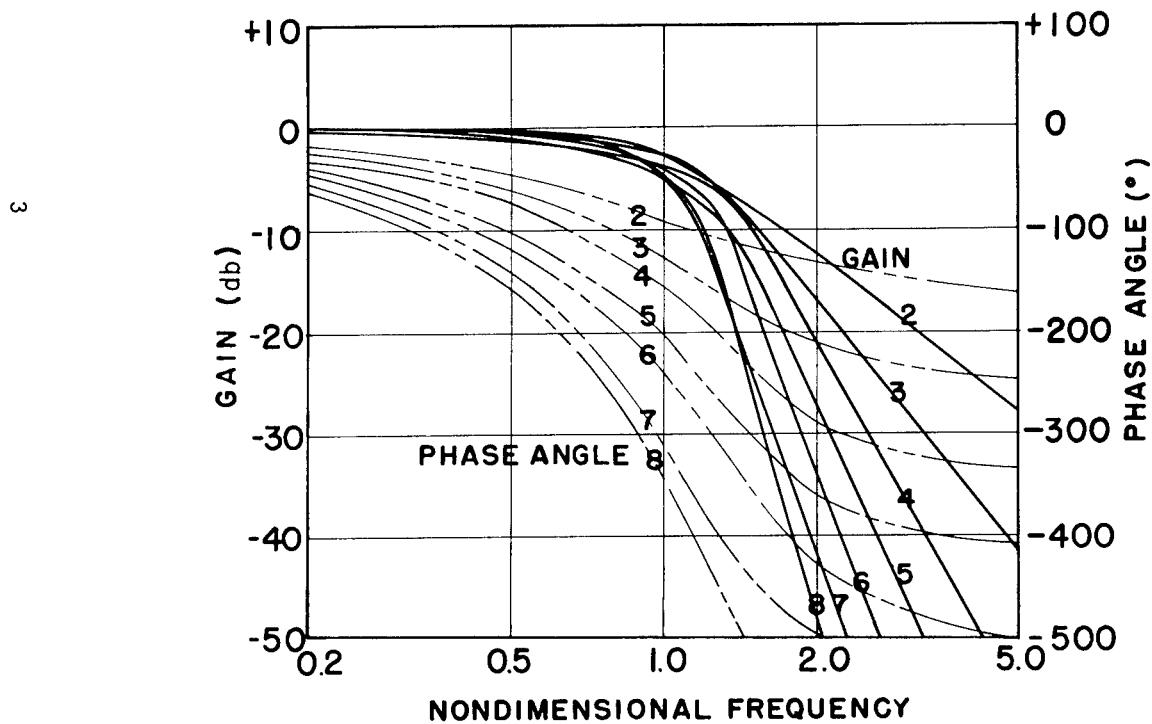
FIGURE 23. FREQUENCY RESPONSE FUNCTIONS OF THE OPTIMUM UNIT-NUMERATOR TRANSFER SYSTEMS, SECOND TO EIGHTH ORDERS.
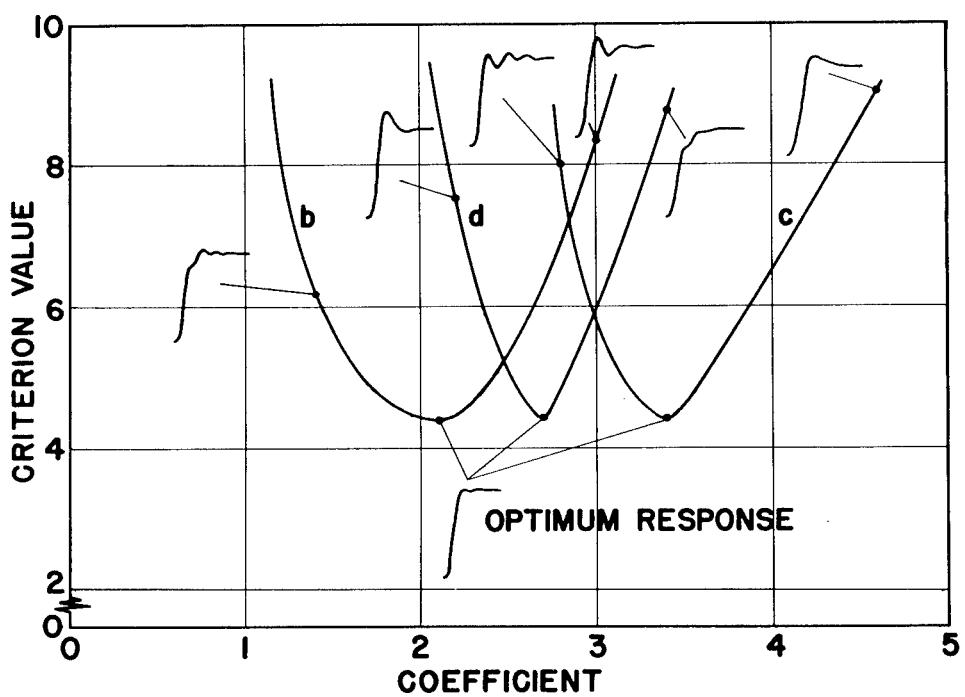
FIGURE 24. SECTIONS THROUGH THE MINIMUM POINT OF THE ITAE SURFACE, FOURTH ORDER UNIT-NUMERATOR SYSTEM.
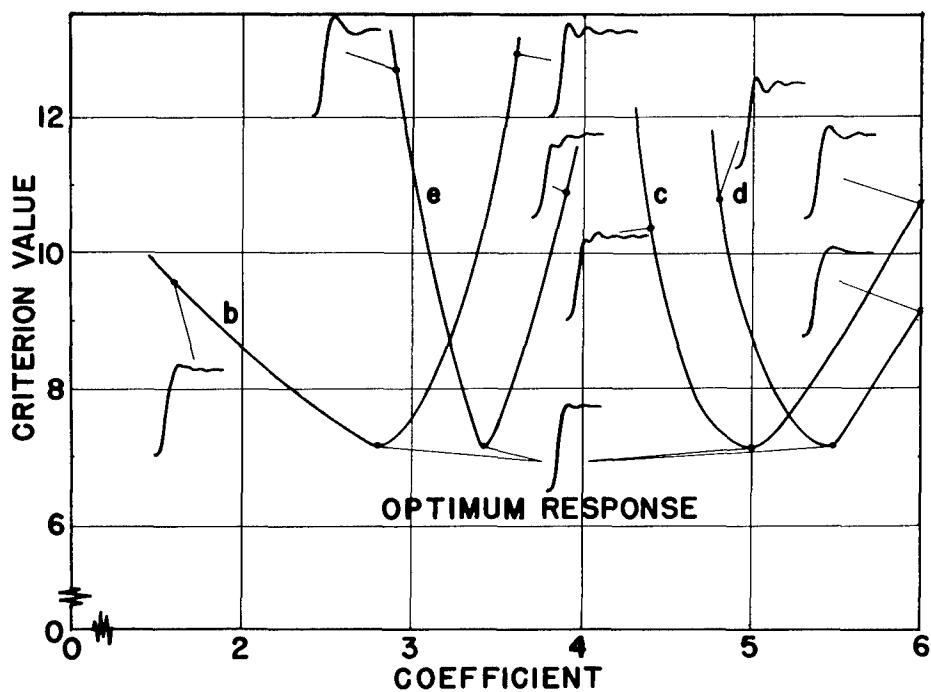
FIGURE 25. SECTIONS THROUGH THE MINIMUM
POINT OF THE ITAE SURFACE, FIFTH ORDER
UNIT-NUMERATOR SYSTEM.
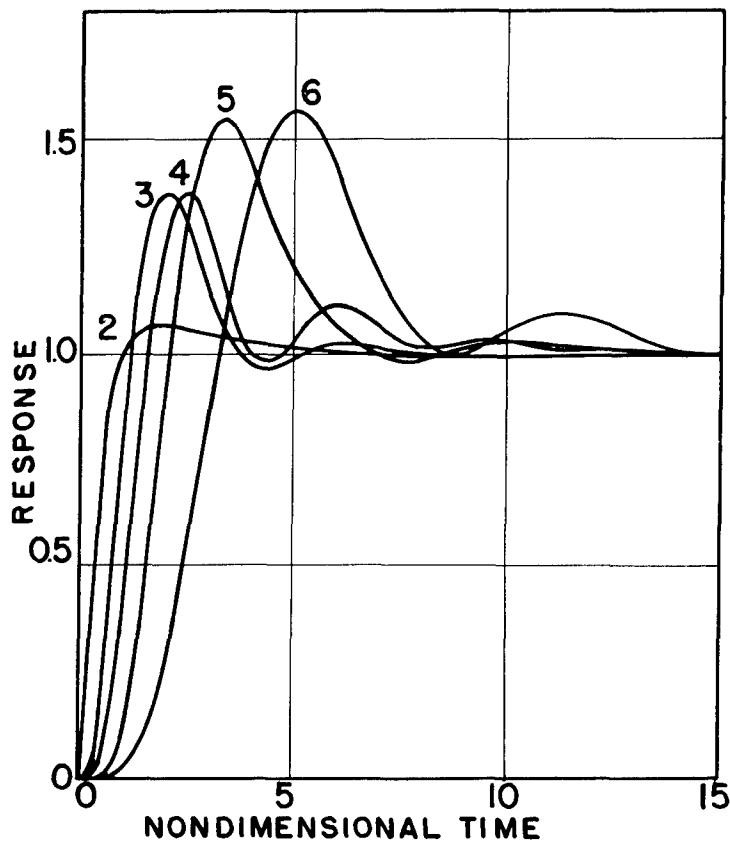
FIGURE 26. STEP FUNCTION RESPONSES OF THE
OPTIMUM ZERO-VELOCITY-ERROR SYSTEMS,
SECOND TO SIXTH ORDERS.
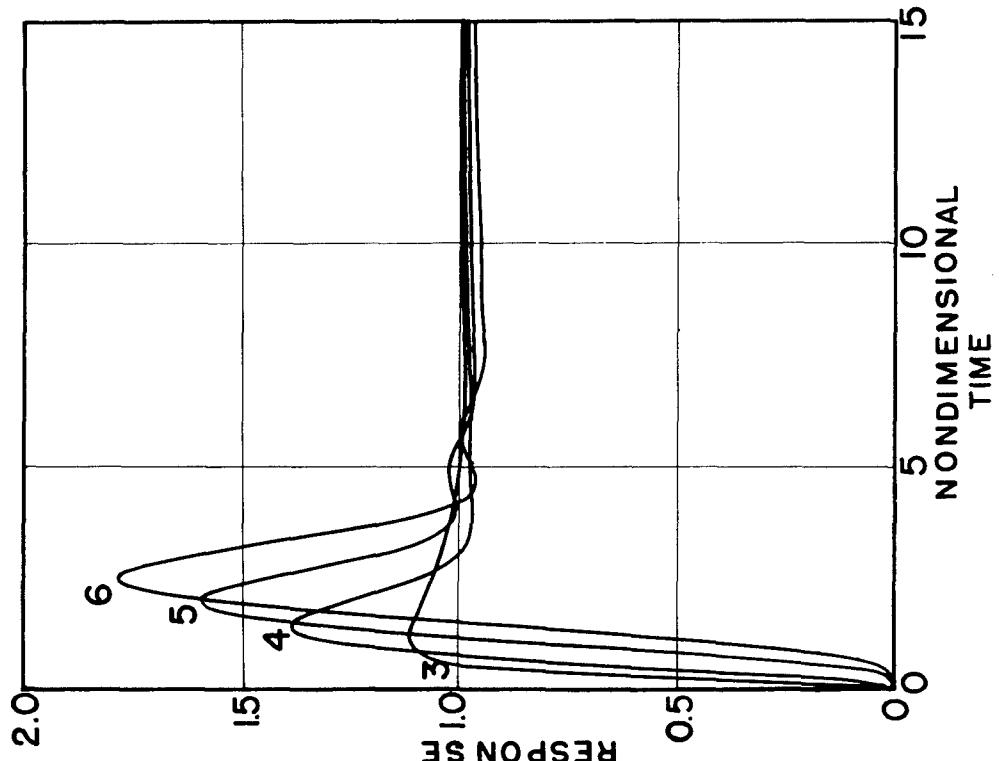
FIGURE 28. STEP FUNCTION RESPONSES OF THE OPTIMUM ZERO-ACCELERATION-ERROR SYSTEMS, THIRD TO SIXTH ORDERS.
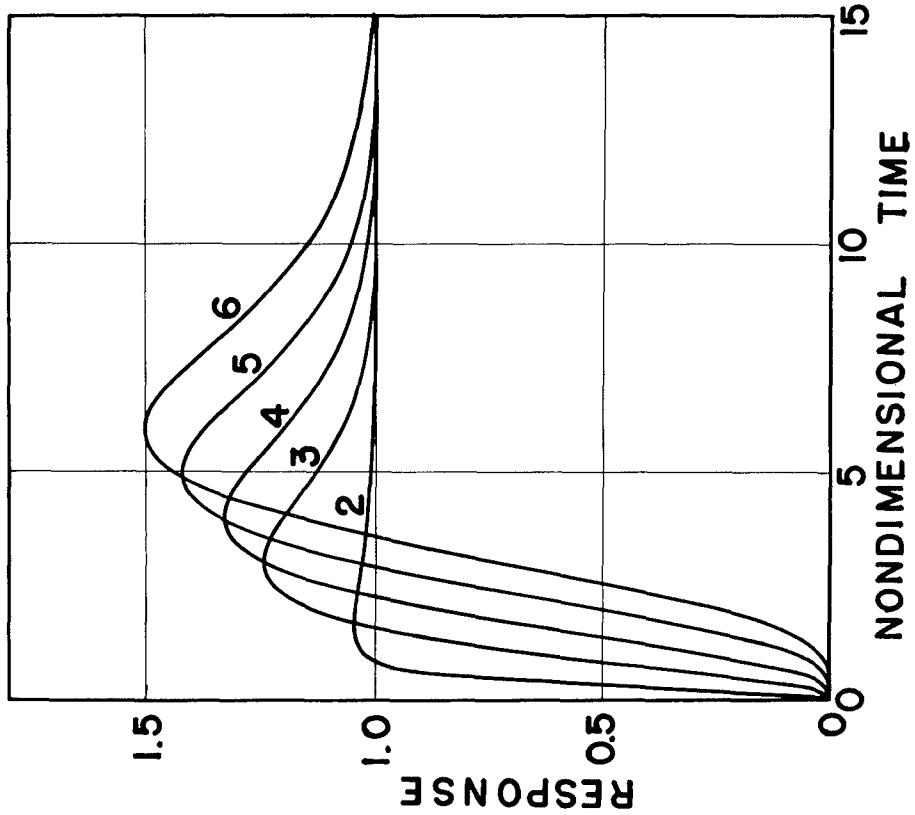
FIGURE 27. STEP FUNCTION RESPONSES OF THE BINOMIAL ZERO-VELOCITY-ERROR SYSTEMS, SECOND TO SIXTH ORDERS.
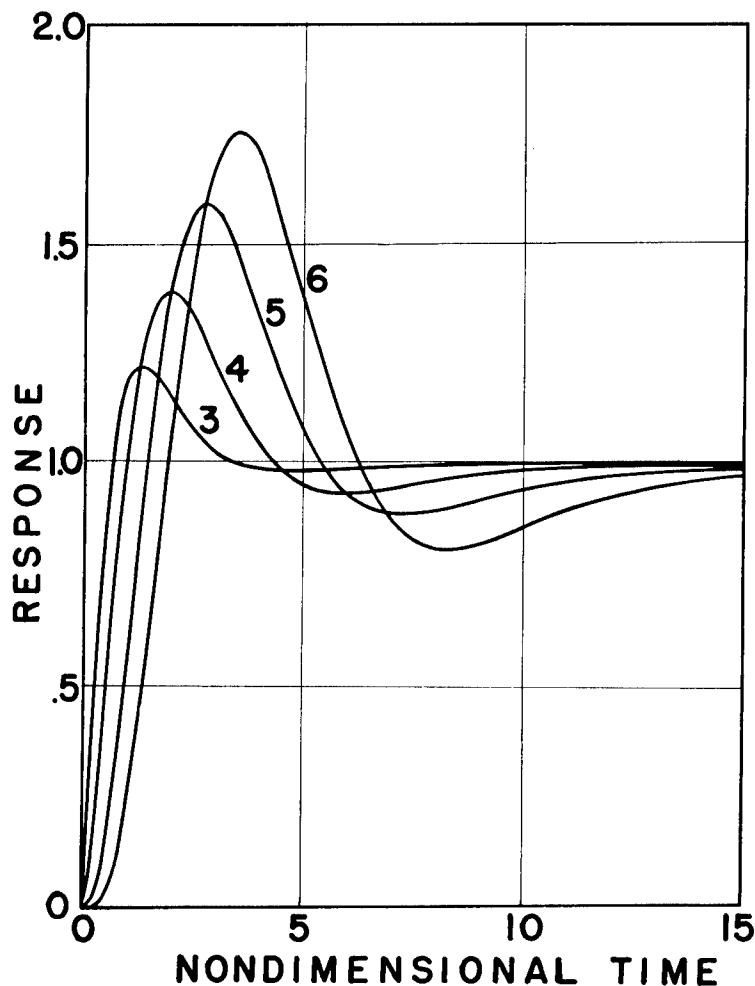
FIGURE 29. STEP FUNCTION RESPONSES OF THE
BINOMIAL ZERO-ACCELERATION-ERROR SYSTEMS,
THIRD TO SIXTH ORDERS.
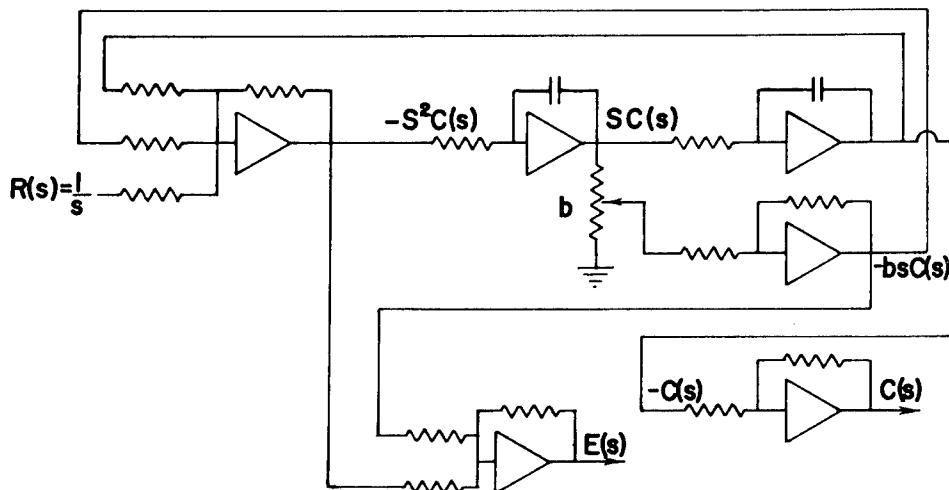
FIGURE 30. ANALOG COMPUTER CIRCUIT FOR
OBTAINING THE RESPONSES OF A SECOND
ORDER TRANSFER SYSTEM.
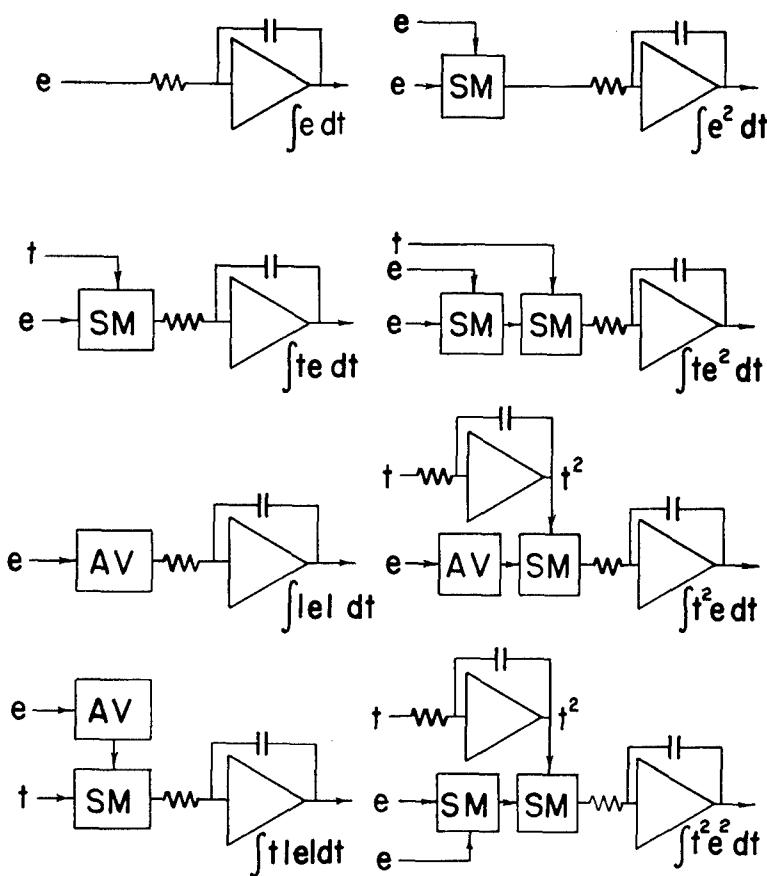
SM = Servo Multiplier   AV = Absolute Value Unit

FIGURE 31. CIRCUITS FOR MECHANIZING PER-
FORMANCE CRITERIA ON AN ELECTRONIC
ANALOG COMPUTER.
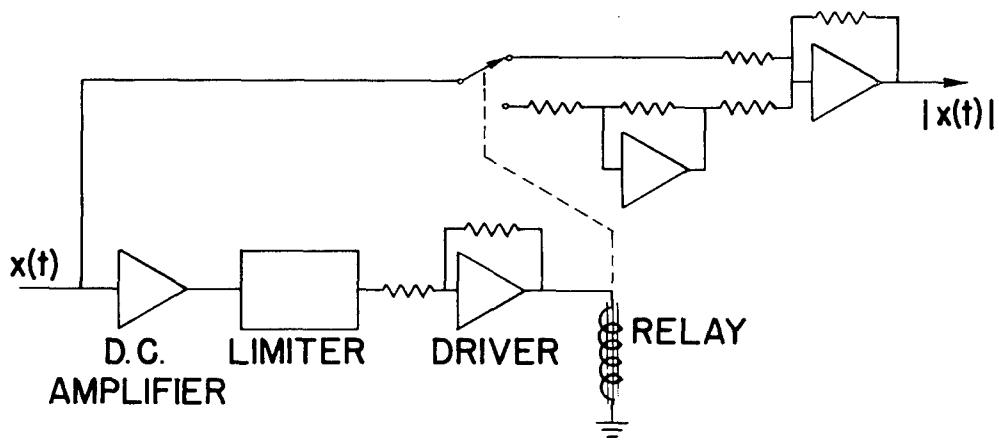
FIGURE 32. THE ABSOLUTE VALUE DEVICE.